The equations of one-dimensional unsteady flame propagation : Existence and uniqueness

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boundary value problem to be well-posed.

Owing to the strong regularizing effect of the heat equation, even a weak solution of the Lagrangian system is continuous. It is then straightforward to come back to the usual Eulerian coordinates, and to prove that similar existence and uniqueness results hold for the original Eulerian system of governing equations.

The paper is organized as follows:

1. Introduction.
2. Governing equations of the flame propagation.
3. Assumptions and main results.
4. Recalling some basic results from semi-group theory.
5. Existence and uniqueness for the combustion variables.
6. Existence and uniqueness for the hydrodynamical variables.
7. Back transformation to the Eulerian variables.
8. Extension to chemically complex flames.

THE EQUATIONS OF

ONE-DIMENSIONAL UNSTEADY FLAME PROPAGATION :

EXISTENCE ET UNIQUENESS

Abstract: This paper deals with the mathematical analysis of a system of partial differential equations describing the time-dependent propagation of a planar flame front within the framework of the well-known isobaric approximation of slow combustion. The problem to be investigated takes the form of a non linear mixed initial-boundary value problem in an infinite one-dimensional domain. We show the existence and uniqueness of weak and classical solutions of this problem, depending on the assumptions on the initial data and on the non linear temperature dependence of the chemical reaction rates. The crucial point lies in the introduction of a Lagrangian space coordinate, which uncouples the reaction-diffusion equations for the combustion variables from the remaining hydrodynamical subsystem. The analysis then uses some classical arguments of functional analysis, such as the application of the theory of linear semigroups to non linear partial differential equations.

Key-words: partial differential equations, reaction-diffusion systems, combustion.

LES EQUATIONS DE LA PROPAGATION

D'UNE FLAMME PLANE INSTATIONNAIRE :

EXISTENCE ET UNICITE

Résumé : On étudie dans ce rapport un système d'équations aux dérivées partielles non linéaires modélisant la propagation instationnaire d'une flamme plane dans un mélange gazeux à faible nombre de Mach. On montre l'existence et l'unicité de solutions faible et classique, selon les hypothèses sur les données initiales et sur la dépendance en température des taux de réaction. L'argument essentiel de l'analyse consiste à introduire une coordonnée Lagrangienne, ce qui découple le système d'équations en deux sous-systèmes: un système non linéaire de réaction-diffusion pour les variables de température et de composition, et un système linéaire pour les variables de vitesse et de pression. Le premier système est résolu en utilisant des outils classiques de l'analyse fonctionnelle (application de la théorie des semi-groupes aux équations aux dérivées partielles non linéaires), et le second système est traité en une deuxième étape.

Mots-clés : équations aux dérivées partielles, systèmes de reaction-diffusion, combustion.

1. INTRODUCTION

The mathematical analysis of systems of ordinary or partial differential equations arising from the theory of gaseous combustion has received an increasing attention in recent years: one can mention for instance several studies of the equations of the stationary planar flame (see [2],[8]) or of the two-dimensional zero Mach number model (see [6]), and in a different domain some mathematical works dealing with the existence and the asymptotic behaviour of the solutions of the Kuramoto-Sivashinsky equation for the flame front instabilities (see [1],[9]).

We present in this paper a new rigorous mathematical result which concerns the time-dependent one-dimensional flame propagation. More precisely, we consider the governing equations of an unsteady planar flame propagating in an infinite channel. These equations, which we recall in Section 2, are written using the classical isobaric approximation for reacting flows in open domains (we first consider a simplified one-step chemical mechanism; the extension to chemically complex flames or to non adiabatic flames is given at the end of the paper). With appropriate hypotheses on the initial data and on the temperature dependence of the reaction rate, we show the global existence and the uniqueness of both weak and classical solutions of the resulting initial-boundary value problem.

The crucial point in our analysis (and in fact the point which restricts our work to the one-dimensional case) lies in the introduction of a Lagrangian space coordinate. This change of coordinates has the effect of decoupling the reaction-diffusion equations for the combustion variables (temperature and mass fraction of the reactant) from the remaining equations for the hydrodynamical variables (density, velocity and pressure). The reactive diffusive system involving the temperature and the mass fraction takes the form of two coupled nonlinear heat equations and is known as the thermo-diffusive model for the flame propagation. This parabolic system of partial differential equations is solved in a first step, using classical tools of nonlinear functional analysis such as semigroups generated by linear operators in functional spaces. The remaining subsystem for the hydrodynamical unknowns is then solved in a second step, the temperature being considered given. This provides the existence and uniqueness of solutions of the Lagrangian system. In particular, the analysis shows that no initial data for the hydrodynamical variables need be given for the initial-

2. GOVERNING EQUATIONS OF THE FLAME PROPAGATION

2.1. Reactive flow equations in one dimension

We are interested in the description of a compressible heat-conducting chemically reacting gaseous mixture with the assumption of a one-dimensional geometry. For the sake of simplicity, we first assume a one-step chemical mechanism $nA \to nB$: the mixture is considered to be made of only two species, the reactant A and the product B. The extension to the case of a chemically complex flame will be investigated in Section 8 below.

The reactive gas flow is then described with the usual variables ρ , u , P , T (denoting respectively the total density, velocity, pressure and temperature of the mixture) and an additional variable for the mixture composition, the mass fraction Y of the reactant A [ρY is the separate density of the reactant and $\rho(1-Y)$ is the density of the product]. The time-dependent flow of this reactive mixture is then described by the following set of equations (see [5],[7],[14]):

$$\begin{cases} \rho_\tau + (\rho u)_\xi = 0 \,, \\ \rho u_\tau + \rho u u_\xi = - P_\xi \,, \\ \rho c_p T_\tau + \rho u c_p T_\xi - (\lambda T_\xi)_\xi = Q\,\omega(Y,T) + P_\tau + uP_\xi \,, \\ \rho Y_\tau + \rho u Y_\xi - (\rho D Y_\xi)_\xi = -m\,\omega(Y,T) \,, \\ \rho T = \dfrac{mP}{R} \,, \end{cases} \tag{2.1}$$

where ξ and τ denote respectively the space and time variables; c_p is the specific heat at constant pressure of the mixture, λ the heat conductivity, D the diffusion coefficient of the reactant A, m its molecular mass, and R is the universal mass constant. The effects of viscosity and gravity are neglected. Lastly, Q (>0) is the amount of energy released by the exothermic chemical reaction per unit mass of the reactant, and $\omega(Y,T)$ is the rate at which this reaction proceeds. From the Arrhenius law and the law of mass action, this reaction rate is given by:

$$\omega(Y,T) = B(T) \left[\frac{\rho Y}{m}\right]^n e^{\frac{-E}{RT}} \,, \tag{2.2}$$

where E is the activation energy of the reaction (a constant), and $B(T)$ is some given function of T (which usually has a polynomial type dependence on T).

2.2. Eulerian form of the flame propagation equations

For writing down the governing equations of the unsteady flame propagation, we will use the so-called "classical approximation of combustion": the flame propagation is essentially a very subsonic, almost isobaric phenomenon. In other words, the Mach number M of the flow is very small and consequently the pressure variations are also small: $P(\xi,\tau) = P_0 + p(\xi,\tau)$, with $\frac{p}{P_0} = O(M^2) \ll 1$. For this reason, we may set $P = P_0 = Constant$ everywhere except in the momentum equation (2.1.b) (see [5],[7] for a more detailed discussion of this approximation). The system (2.1) then reduces to:

$$\begin{cases} \rho_\tau + (\rho u)_\xi = 0 \, , \\ \rho c_p T_\tau + \rho u c_p T_\xi - (\lambda T_\xi)_\xi = Q\,\omega(Y,T) \, , \\ \rho Y_\tau + \rho u Y_\xi - (\rho D Y_\xi)_\xi = -m\,\omega(Y,T) \, , \\ \rho T = \dfrac{m P_0}{R} \, ; \end{cases} \tag{2.3}$$

$$\rho u_\tau + \rho u u_\xi = - p_\xi \, . \tag{2.4}$$

Some authors use the system (2.3) alone, replacing the momentum equation (2.4) by $P = P_0$ (see [14]). This is legitimate in one spatial dimension since the only role of the relation (2.4) is the calculation of the small pressure variation p. But this simplification is no more valid when the space dimension N is higher than one, since it eliminates N scalar momentum equations and only one variable p. For this reason we will mainly consider the full system (2.3)-(2.4).

The flame propagation equations (2.3)-(2.4) will be investigated with the following upstream and downstream boundary conditions:

$$Y(-\infty,t) = Y_u \, , \; T(-\infty,t) = T_u \, , \; u(-\infty,t) = u^0 \, , \; p(-\infty,t) = 0 \, , \tag{2.5}$$

(where $Y_u > 0$, $T_u > 0$, $u^0 \in \mathbb{R}$ are given constants) in the fresh mixture, and:

$$Y(+\infty,t) = 0 \, , \; T(+\infty,t) = T_b = T_u + \frac{Q}{c_p}\frac{Y_u}{m} \, . \tag{2.6}$$

in the burnt gases.

2.3. Lagrangian form of the governing equations

From now on we will assume that the Lewis number $Le = \dfrac{\lambda}{\rho c_p D}$, and the specific heat c_p are constant. We will also assume that the thermal conductivity of the mixture λ is proportional to the temperature T; this additional assumption will be discussed below, after the derivation of the Lagrangian equations.

We now derive an alternate formulation of the governing equations (2.3)-(2.4) using the usual mass-weighted Lagrangian coordinate:

$$x = \int_{\xi(0,t)}^{\xi(x,t)} \rho(\xi',t)\ d\xi' . \tag{2.7}$$

Although the use of this transformation is classical, we detail the calculation for sake of completeness. Let us define a Lagrangian coordinate (i.e. a variable whose value, defined at time $\tau = 0$, remains constant during the flow for each fluid particle) by setting:

$$x = \int_0^{\xi} \rho(\xi',0)\ d\xi' .$$

We also set $t = \tau$. Then $x(\xi,\tau)$ represents the Lagrangian coordinate of the particle which is located at the abcissa ξ at time τ and the last relation is to be read as $x(\xi,0) = \int_0^{\xi} \rho(\xi',0)\ d\xi'$. Inversely, $\xi(x,t)$ is the position at time t of the fluid particle whose Lagrangian coordinate is x. Therefore we have, by definition:

$$\xi_t = u \ , \ \text{or} \ \frac{\partial}{\partial t}\xi(x,t) = u[\xi(x,t),t] .$$

We can then write:

$$\frac{d}{dt}\left[\int_{\xi(0,t)}^{\xi(x,t)} \rho(\xi',t)\ d\xi' \right] = \frac{\partial \xi}{\partial t}(x,t)\rho[\xi(x,t),t] - \frac{\partial \xi}{\partial t}(0,t)\rho[\xi(0,t),t]$$

$$+ \int_{\xi(0,t)}^{\xi(x,t)} \frac{\partial \rho}{\partial t}(\xi',t)\ d\xi' ,$$

$$= (\rho u)[\xi(x,t),t] - (\rho u)[\xi(0,t),t] + \int_{\xi(0,t)}^{\xi(x,t)} \frac{\partial \rho}{\partial t}(\xi',t)\ d\xi' ,$$

$$= \int_{\xi(0,t)}^{\xi(x,t)} [\rho_t + (\rho u)_{\xi'}](\xi',t)\ d\xi' = 0 ,$$

whence:

$$\int_{\xi(0,t)}^{\xi(x,t)} \rho(\xi',t)\ d\xi' = \int_{\xi(0,0)}^{\xi(x,0)} \rho(\xi',0)\ d\xi' = x ,$$

which is exactly (2.7).

Differentiating (2.7) with respect to x gives:

$$1 = \rho \xi_x \text{ , or } \frac{\partial}{\partial x}\xi(x,t) = \frac{1}{\rho[\xi(x,t),t]} .$$

We then have in matrix form (writing simply $u(x,t)$ for $u[\xi(x,t),t]$):

$$\begin{bmatrix} \xi_x & \xi_t \\ \tau_x & \tau_t \end{bmatrix} = \begin{bmatrix} \rho^{-1} & u \\ 0 & 1 \end{bmatrix} ,$$

which implies:

$$\begin{bmatrix} x_\xi & x_\tau \\ t_\xi & t_\tau \end{bmatrix} = \begin{bmatrix} \rho & -\rho u \\ 0 & 1 \end{bmatrix} . \tag{2.8}$$

REMARK 2.1: The mass balance equation (2.3.a) has been crucial for introducing the new variable x. This amounts to noticing that a variable X satisfying $X_\xi = \rho$, $X_\tau = -\rho u$ [i.e. (2.8)] could have been introduced directly, since (2.3.a) insures that $\frac{\partial}{\partial \tau}(X_\xi) = \frac{\partial}{\partial \xi}(X_\tau)$. ▪

We can now derive the Lagrangian form of the flame propagation equations. For any quantity F we have $F_\tau = F_t - \rho u F_x$, $F_\xi = \rho F_x$, and the system (2.3)-(2.4) becomes:

$$\begin{cases} \rho_t + \rho^2 u_x = 0 \,, \\ u_t + p_x = 0 \,, \\ T_t = \dfrac{Q}{c_p}\dfrac{\omega}{\rho} + \dfrac{1}{c_p}(\lambda\rho T_x)_x \,, \\ Y_t = -\, m\dfrac{\omega}{\rho} + (\rho^2 D Y_x)_x \,, \\ \rho T = \dfrac{mP_0}{R} \,. \end{cases} \tag{2.9}$$

To nondimensionalize these equations, we refer the mass fraction to $Y_0 = Y_u$, the temperature to $T_0 = T_b - T_u = \frac{Q}{c_p}\frac{Y_u}{m}$, the density to $\rho_0 = \frac{mP_0}{RT_0}$. Denoting $(\lambda\rho)_0$ the constant value of $\lambda\rho = \frac{mP_0}{R}\frac{\lambda}{T}$, we relate the time unit t_0 and the "Lagrangian unit" x_0 by: $x_0^2 = \frac{t_0(\lambda\rho)_0}{c_p}$. The velocity is then referred to $u_0 = \frac{x_0}{\rho_0 t_0}$ and the pressure variation to $p_0 = \rho_0 u_0^2$.

Setting $\Theta = T - T_u$ and denoting by $\hat{\Theta}$, $\hat{Y}$, $\hat{\rho}$, $\hat{u}$, $\hat{p}$ the nondimensionalized variables, we obtain the following expressions for the Lagrangian equations (2.9) and boundary conditions (2.5)-(2.6):

$$\begin{cases} \hat{\Theta}_t = \hat{\Theta}_{xx} + \Omega(\hat{Y},\hat{\Theta}) \,, \\ \hat{Y}_t = \dfrac{1}{Le}\hat{Y}_{xx} - \Omega(\hat{Y},\hat{\Theta}) \,; \end{cases} \tag{2.10}$$

$$\begin{cases} (\hat{\Theta} + \alpha)\hat{\rho} = 1 \,, \\ \hat{u}_x = \hat{\Theta}_t \,, \\ \hat{u}_t + \hat{p}_x = 0 \,; \end{cases} \tag{2.11}$$

$$\begin{cases} \hat{\Theta}(-\infty,t) = 0 \;\;, \hat{\Theta}(+\infty,t) = 1 \,, \\ \hat{Y}(-\infty,t) = 1 \;\;, \hat{Y}(+\infty,t) = 0 \,, \\ \hat{u}(-\infty,t) = \hat{u}^0 \;\;, \hat{p}(-\infty,t) = 0 \,. \end{cases} \tag{2.12}$$

where x and t now represent the nondimensionalized Lagrangian coordinates, $\alpha = \frac{T_u}{T_b - T_u}$ is a nondimensional heat release parameter, and

$\Omega(\hat{Y},\hat{\Theta}) = \frac{Q}{c_p}\frac{R}{mP_0}t_0\frac{\omega}{\hat{\rho}}$ is the normalized reaction rate. In the sequel, we will assume using (2.2) that Ω is given by:

$$\Omega(\hat{Y},\hat{\Theta}) = \hat{Y}^n f(\hat{\Theta}),$$

where f is a positive continuous function satisfying $f(0) = 0$.

REMARK 2.2: The assumption $f(0) = 0$ is not fulfilled in view of the expression (2.2) of the reaction rate ω since $e^{\frac{-E}{RT_u}} \neq 0$. This is the well known "cold boundary difficulty", on which a lot has already been said (see [5]). Let us just point out that this hypothesis is necessary for the mathematical problem (2.10)-(2.12) to be well-posed. ▪

It should be emphasized here that the use of the Lagrangian coordinate (2.7) uncouples the equations (2.10) for the combustion field $(\hat{\Theta},\hat{Y})$ (which take the form of a purely diffusive reactive system) from the equations (2.11) for the hydrodynamical variables $(\hat{\rho},\hat{u},\hat{p})$. Moreover, the form of these hydrodynamical equations leads to think that no initial data for the density, velocity or pressure is needed to determine the profiles of these variables at positive time values: these hydrodynamical profiles $\hat{\rho}(.,t)$, $\hat{u}(.,t)$, $\hat{p}(.,t)$ for $t > 0$ only depend of the temperature profiles $\hat{\Theta}(.,t')$ for $t' \geqq 0$: we first have to study the nonlinear parabolic system (2.10), and (2.11) will be investigated in a second step.

REMARK 2.3: The assumption $\lambda\rho = Constant$, or $\frac{\lambda}{T} = Constant$ only affects the expression of the diffusive terms in the temperature and mass fraction equations: these terms take the form $\hat{\Theta}_{xx}$ and $\frac{1}{Le}\hat{Y}_{xx}$ instead of $[(\lambda\rho)T_x]_x$ and $\frac{1}{Le}[(\lambda\rho)Y_x]_x$ where, in complete generality, $\lambda\rho$ is a function of T and Y. Nevertheless, it can be noticed that this hypothesis (which is rather classical in combustion theory; see [11]) does not change the preceding remarks about the nature of the Lagrangian system (2.10)-(2.11). We hope to extend our mathematical analysis to the case of a non constant $\frac{\lambda}{T}$ ratio in a forthcoming paper. ▪

REMARK 2.4: In the classical nondimensionalization of the Eulerian equations (2.3)-(2.4) (see [5],[7]), the length and time scales ξ_0 and $\tau_0 = t_0$ are related to the thermal diffusion coefficient $(\frac{\lambda}{\rho c_p})_0 = \frac{(\lambda\rho)_0}{\rho_0^2 c_p}$ and to the velocity unit u_0 by the identities:

$$\frac{\xi_0^2}{\tau_0} = \frac{(\lambda\rho)_0}{\rho_0^2 c_p} \quad \text{and} \quad u_0 = \frac{\xi_0}{\tau_0} .$$

In our case, the units used above to nondimensionalize the equations (2.9) have essentially been chosen in order to simplify the Lagrangian system (2.10)-(2.11), which will play a crucial role in the sequel. Therefore, these units are not quite usual, and the above relations are replaced by:

$$\frac{\xi_0^2}{\tau_0} \left[\int_0^1 \hat{\rho}(\zeta,0)\, d\zeta \right]^2 = \frac{(\lambda\rho)_0}{\rho_0^2 c_p} \quad \text{and} \quad u_0 = \frac{\xi_0}{\tau_0} \int_0^1 \hat{\rho}(\zeta,0)\, d\zeta .$$

since $x_0 = \int_0^{\xi_0} \rho(\xi',0)\ d\xi' = \rho_0 \xi_0 \int_0^1 \hat{\rho}(\zeta,0)\, d\zeta$. ▪

3. ASSUMPTIONS AND MAIN RESULTS

3.1. Statement of the problem

The aim of this paper is to investigate the following version of (2.10)-(2.12):

$$\left\{\begin{array}{l} \Theta_t - \Theta_{xx} = \Omega(Y,\Theta) = Y^n f(\Theta)\,, \\ Y_t - \dfrac{Y_{xx}}{Le} = -\,\Omega(Y,\Theta)\,, \\ (\Theta + \alpha)\rho = 1\,, \\ u_x = \Theta_t\,, \qquad\qquad \text{for } x \in \mathbb{R}\,,\ t \in \mathbb{R}_+\,; \\ \Theta(x,0) = \Theta_0(x)\,,\ Y(x,0) = Y_0(x)\,; \end{array}\right. \tag{3.1}$$

$$\left\{\begin{array}{l} \Theta(-\infty,t) = 0\,,\ \Theta(+\infty,t) = 1\,, \\ Y(-\infty,t) = 1\,,\ Y(+\infty,t) = 0\,, \\ u(-\infty,t) = u^0\,; \end{array}\right. \tag{3.2}$$

$$\left\{\begin{array}{l} u_t + p_x = 0\,, \\ p(-\infty,t) = 0\,. \end{array}\right. \tag{3.3}$$

We will also study the corresponding normalized Eulerian formulation in conservative form:

$$\left\{\begin{array}{l} \rho_\tau + (\rho u)_\xi = 0\,, \\ (\rho u)_\tau + (\rho u^2)_\xi + p_\xi = 0\,, \\ (\rho\Theta)_\tau + (\rho u\Theta)_\xi - (\dfrac{\Theta_\xi}{\rho})_\xi = \rho\Omega(Y,\Theta)\,, \\ (\rho Y)_\tau + (\rho u Y)_\xi - \dfrac{1}{Le}(\dfrac{Y_\xi}{\rho})_\xi = -\,\rho\Omega(Y,\Theta)\,, \\ (\Theta + \alpha)\rho = 1\,, \qquad\qquad \text{for } x \in \mathbb{R}\,,\ t \in \mathbb{R}_+\,; \\ \Theta(x,0) = \Theta_0(x)\,,\ Y(x,0) = Y_0(x)\,. \end{array}\right. \tag{3.4}$$

$$\left\{\begin{array}{l} \Theta(-\infty,t) = 0\,,\ \Theta(+\infty,t) = 1\,, \\ Y(-\infty,t) = 1\,,\ Y(+\infty,t) = 0\,, \\ u(-\infty,t) = u^0\,,\ p(-\infty,t) = 0\,. \end{array}\right. \tag{3.5}$$

It can be noticed here that initial data are prescribed only for the temperature and mass fraction (Θ,Y) and not for the hydrodynamical unknowns (ρ,u,p).

For the investigation of these two problems, we will mainly focus on two types of solutions, which we define precisely below:

DEFINITION 3.1:

(Θ, Y, ρ, u, p) is a *weak solution* of problem (3.1)-(3.3) if the three following properties hold:

(1) $(\Theta, Y, \rho, u, p) \in [L^\infty_{loc}(\mathbb{R} \times \mathbb{R}_+)]^5$ and (Θ, Y, ρ, u, p) is a solution of (3.1)-(3.3.a) in the sense of the distributions:

$$\begin{cases} \int_{\mathbb{R}\times\mathbb{R}_+} [-\Theta\eta_t + \Theta\eta_{xx} - \Omega\eta] = \int_{\mathbb{R}} \Theta_0\eta(.,0)\,, \\ \int_{\mathbb{R}\times\mathbb{R}_+} [-Y\eta_t + \frac{1}{Le}Y\eta_{xx} + \Omega\eta] = \int_{\mathbb{R}} Y_0\eta(.,0)\,, \\ \int_{\mathbb{R}\times\mathbb{R}_+} [(\Theta + \alpha)\rho\eta - \eta] = 0\,, \\ \int_{\mathbb{R}\times\mathbb{R}_+} [u\eta_x - \Theta\eta_t] = \int_{\mathbb{R}} \Theta_0\eta(.,0)\,, \\ \int_{\mathbb{R}\times\mathbb{R}_+} -[u\eta_t + p\eta_x] = \int_{\mathbb{R}} u(.,0)\eta(.,0)\,, \qquad \text{for any } \eta \in D(\mathbb{R}\times\mathbb{R}_+). \end{cases}$$

(2) The boundary conditions (3.2) hold in the classical sense for $t > 0$ and (3.3.b) holds in the following weak sense:

$$\forall\, t > 0\,,\ \exists\, p_1 \in L^2(\mathbb{R})\,,\ \lim_{x\to-\infty} [p(x,t) - p_1(x)] = 0\,. \tag{3.6}$$

(3) The following inequalities (which are necessary from a physical standpoint) hold:

$$\Theta(x,t) \ge 0\,,\ 0 \le Y(x,t) \le 1 \text{ a.e. on } \mathbb{R}\times\mathbb{R}_+\,.$$

Moreover, $\Theta \in L^\infty_{loc}(\mathbb{R}_+, L^\infty(\mathbb{R}))$. ▪

DEFINITION 3.2:

A *weak solution* (Θ, Y, ρ, u, p) of problem (3.1)-(3.3) is a *smooth solution* if and only if:

(1) All the functions and all the partial derivatives appearing in the equations (3.1) and (3.3.a) are continuous with respect to both variables x and t on $\mathbb{R}\times\mathbb{R}_+$.

(2) The boundary conditions (3.2) and (3.3.b) are fulfilled in the classical sense for $t \geqq 0$. ▪

Similar definitions hold for the solutions of (3.4)-(3.5).

3.2. Assumptions and notations

Before stating the main hypotheses which will be used for investigating the two above problems, we need to introduce two functions γ and γ_1 of $C^\infty(\mathbb{R})$ satisfying:

$$\begin{cases} \gamma = 0 \ \text{ on } (-\infty,-1]\,, \ 0 \leqq \gamma \leqq 1 \ \text{ on } [-1,1]\,, \ \gamma = 1 \ \text{ on } [1,+\infty)\,; \\ \gamma_1 = 1 - \gamma\,. \end{cases} \tag{3.7}$$

We will set:

$$\begin{cases} \varphi_0(x) = \Theta_0(x) - \gamma(x)\,, \\ \psi_0(x) = Y_0(x) - \gamma_1(x)\,. \end{cases} \tag{3.8}$$

The following assumptions will be used in the theorems stated below:

$$\left.\begin{array}{c} \varphi_0 \in L^2(\mathbb{R})\,, \ \psi_0 \in L^2(\mathbb{R})\,; \\[2mm] \begin{cases} \Theta_0 \in L^\infty(\mathbb{R})\,, \ \Theta_0(x) \geqq 0 \ \text{ a.e.}\,, \\ Y_0(x) \in [0,1] \ \text{ a.e.}\,; \end{cases} \\[2mm] Le > 0 \ \text{ and } n \in \mathbb{N}^* \text{ are given}\,; \\[2mm] \begin{cases} f \in C(\mathbb{R}_+,\mathbb{R}_+)\,, \ f(0) = 0\,, \\ \forall\, \vartheta > 0\,, f \text{ is Lipschitz-continuous on } [0,\vartheta]\,. \end{cases} \end{array}\right\} \tag{3.9}$$

Moreover we will sometimes need some of the following more technical hypotheses:

$$\exists\, C_f > 0\,, \ \forall\, \vartheta \in \mathbb{R}_+\,, \ |f(\vartheta)| \leqq C_f\,|\vartheta|\,; \tag{3.10}$$

$$\varphi_0 \in H^2(\mathbb{R})\,, \ \psi_0 \in H^2(\mathbb{R})\,; \tag{3.11}$$

$$\begin{cases} f \in C^1(\mathbb{R}_+,\mathbb{R}_+)\,; \\ \exists\, \beta > \dfrac{1}{2}\,, \ \lim\limits_{\vartheta\to 0} \dfrac{|f'(\vartheta)|}{\vartheta^\beta} < +\infty\,; \end{cases} \tag{3.12}$$

$$\exists\, \mu > \frac{3}{2\beta}\,,\ \sup_{x\in\mathbb{R}_-} \Theta_0(x)\,|x|^{\mu} < +\infty\,. \tag{3.13}$$

$$\varphi_0 \in H^4(\mathbb{R})\,,\ \psi_0 \in H^4(\mathbb{R})\,; \tag{3.14}$$

$$\begin{cases} f \in C^2(\mathbb{R}_+,\mathbb{R}_+)\,, \\ \forall\, \vartheta > 0\,,\ f_{xx} \text{ is Lipschitz-continuous on } [0,\vartheta]\,; \end{cases} \tag{3.15}$$

From now on, we will denote $L^p = L^p(\mathbb{R})$, for $p \in [1,+\infty)$, and $\|\varphi\|_p = \|\varphi\|_{L^p}$ or $\|(\varphi,\psi)\|_p = \max\,(\|\varphi\|_{L^p}\,, \|\psi\|_{L^p})$. Furthermore, for $m \in \mathbb{N}^*$, we set $H^m = H^m(\mathbb{R}) = W^{m,2}(\mathbb{R})$.

3.3. Results concerning the Lagrangian formulation

The first of our theorems deals with the problem (3.1)-(3.2) without the pressure variable:

THEOREM 3.3:

Assume that the hypotheses (3.9) and (3.10) hold. Then there exists a unique *weak solution* (Θ,Y,ρ,u) of (3.1)-(3.2) in $\mathbb{R} \times \mathbb{R}_+$ satisfying:

$$\Theta-\gamma,\ Y-\gamma_1\ \in C(\mathbb{R}_+,L^2)\,. \tag{3.16}$$

Furthermore, this solution satisfies:

$$\begin{cases} \Theta,\, Y,\, \rho\ \in C(\mathbb{R}_+,L^\infty) \cap C(\mathbb{R}_+^*,C^1(\mathbb{R}))\,, \\ \Theta-\gamma,\ Y-\gamma_1 \in C^1(\mathbb{R}_+^*,L^2)\,, \\ u\ \in C(\mathbb{R}_+^*,C(\mathbb{R})\cap L^\infty)\,. \blacksquare \end{cases} \tag{3.17}$$

Concerning the complete system (3.1)-(3.3), we have the two following results:

THEOREM 3.4: Assume that the hypotheses (3.9)-(3.13) hold. Then there exists a unique *weak solution* (Θ,Y,ρ,u,p) of (3.1)-(3.3) in $\mathbb{R} \times \mathbb{R}_+$ satisfying (3.16). Moreover this solution satisfies (3.17). ▪

THEOREM 3.5: Assume that all the hypotheses (3.9) to (3.15) hold. Then there exists a unique *smooth solution* of (3.1)-(3.3) in $\mathbb{R} \times \mathbb{R}_+$. This solution satisfies:

$$\begin{cases} \Theta, Y, \rho \in C(\mathbb{R}_+, C^3(\mathbb{R})) \cap C^1(\mathbb{R}_+, C^1(\mathbb{R})) , \\ u \in C(\mathbb{R}_+, C^2(\mathbb{R})) \cap C^1(\mathbb{R}_+, C(\mathbb{R})) . \blacksquare \end{cases} \tag{3.18}$$

These three theorems will be proved in Sections 5 and 6 below.

3.4. Results concerning the Eulerian formulation

Analogous results hold for the Eulerian problem (3.4)-(3.5):

THEOREM 3.6: Assume that the hypotheses (3.9)-(3.13) hold. Then there exists a unique *weak solution* (Θ, Y, ρ, u, p) of (3.4)-(3.5) in $\mathbb{R} \times \mathbb{R}_+$ satisfying (3.16) and:

$$u, \rho \in C(\mathbb{R} \times \mathbb{R}_+, \mathbb{R}) .$$

Moreover, this solution satisfies (3.17). ▪

THEOREM 3.7: Assume that all the hypotheses (3.9) to (3.15) hold. Then there exists a unique *smooth solution* of (3.4)-(3.5) in $\mathbb{R} \times \mathbb{R}_+$. This solution satisfies (3.18). ▪

The proof of these two last results is detailed in Section 7.

4. RECALLING SOME BASIC RESULTS FROM SEMI-GROUP THEORY.

In this section, we briefly recall some classical results from functional analysis which will be needed in the following sections. We refer the reader to [3],[4],[10],[15] for more details and for the proofs of these results.

4.1. Semigroups of linear operators

Let us first recall some basic definitions and results about maximal monotone linear operators:

Let H be a real Hilbert space and A be an unbounded linear operator defined on the subspace $D(A) \subset H$. The operator A is said to be maximal monotone if and only if:

$$\begin{cases} \forall\, u \in D(A), \ (Au,u) \geq 0 \,; \\ \forall\, v \in H\,, \ \exists\, u \in D(A)\,, \ v = u + Au \,. \blacksquare \end{cases} \tag{4.1}$$

The basic property is the theorem of Hille-Yosida:

THEOREM 4.1: (Hille-Yosida)

Let H be a real Hilbert space and A be a maximal monotone linear operator defined on the subspace $D(A) \subset H$. For $u_0 \in D(A)$, the problem:

$$\begin{cases} \dfrac{du}{dt} + Au = 0 \ \text{ for } t \geq 0\,, \\ u(0) = u_0\,. \end{cases} \tag{4.2}$$

has a unique solution in $C(\mathbb{R}_+, D(A)) \cap C^1(\mathbb{R}_+, H)$. ▪

Let $u(t)$ be the solution of (4.2) for $t \geq 0$; we set $u(t) = R(t)u_0$, where $R(t)$ is a linear operator from $D(A)$ into H. Since it follows from (4.1) that $D(A)$ is a dense subspace of H, we can extend $R(t)$ to the whole space H; the resulting operator, which we still denote by $R(t)$, is (by definition) the linear semigroup generated by $-A$.

Let us finally recall that a maximal monotone operator A is self-adjoint if and only if: $\forall\,(u,v) \in D(A)^2$, $(Au,v) = (u,Av)$. ▪

4.2. Nonlinear equations

We are going to consider some problems of the form:

$$\begin{cases} \dfrac{du}{dt} + Au = F(u) \ \text{ for } t \geq 0 , \\ u(0) = u_0 ; \end{cases} \tag{4.3}$$

where A is a linear self-adjoint maximal monotone operator, $u_0 \in H$ and $F \in C(H,H)$. Before stating results about the existence of solution of this problem, we precise which type of solution will be considered:

DEFINITION 4.2:

u is a *classical solution* of (4.3) on an interval $[0,T)$ if and only if u satisfies (4.3) in the classical sense, i.e. with:

$$u \in C^1([0,T),H) \cap C([0,T),D(A)) .$$

u is a *weak solution* of (4.3) on $[0,T)$ if and only if $u \in C([0,T),H)$ and:

$$\forall\ t \in [0,T) ,\ u(t) = R(t)u_0 + \int_0^t R(t-s)\, F[u(s)]\, ds\ . \blacksquare \tag{4.4}$$

We can then state the two following theorems:

THEOREM 4.3:

Let H be a real Hilbert space and A be a linear self-adjoint maximal monotone operator defined on the subspace $D(A) \subset H$. Assume that F is a Lipschitz-continuous mapping from H into itself. Then for any $u_0 \in H$, there exists a unique *weak solution* of (4.3) in $\mathbb{R}_+$ and this solution u is classical on $\mathbb{R}_+^*$.

Moreover, if $u_0 \in D(A)$, then u is a *classical solution* on $\mathbb{R}_+$. $\blacksquare$

THEOREM 4.4:

Let H be a real Hilbert space and A be a linear self-adjoint maximal monotone operator defined on the subspace $D(A) \subset H$. Assume that F is a Lipschitz-continuous mapping from any bounded subset of H into H. Then for any $u_0 \in H$, there exists $T_{\max} > 0$ such that a unique *weak solution* of (4.3) exists on $[0,T_{\max})$; this solution u is classical on $(0,T_{\max})$ and the following alternative holds:

$$\begin{cases} \text{Either:} \quad T_{\max} = +\infty, \\ \text{Or:} \quad \lim\limits_{t \to T_{\max}} \|u(t)\|_H = +\infty. \end{cases}$$

Moreover, if $u_0 \in D(A)$, then u is a *classical solution* on $[0,T_{\max})$. ▪

4.3. Application to the heat equation

We now consider the case $H = L^2$, and the operator:

$$A : \left| \begin{array}{l} D(A) = H^2 \to L^2, \\ \quad \varphi \to -\varphi_{xx}. \end{array} \right.$$

Problem (4.3) then becomes a non linear heat equation; Theorems 4.3 and 4.4 apply to this case because of the following lemma:

LEMMA 4.5:

A is a self-adjoint maximal monotone operator. ▪

Let $S(t)$ be the semigroup generated by $-A$; the following properties of this semigroup will be useful in the sequel:

LEMMA 4.6:

The following properties hold for the semigroup $S(t)$:

$$\forall\, p \in [1,\infty),\ \forall\, \varphi \in L^2 \cap L^p,\ \forall\, t \in \mathbb{R}_+,\ \|S(t)\varphi\|_{L^p} \leq \|\varphi\|_{L^p};$$

$$\forall\, \varphi \in L^2 \cap L^\infty,\ \forall\, t \in \mathbb{R}_+,\ S(t)\varphi \in C(\mathbb{R}_+,L^\infty). \ ▪$$

LEMMA 4.7:

Let $u_0 \in L^2$. The following explicit expression holds for $S(t)u_0$:

$$[S(t)u_0](x) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} u_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy \ . \blacksquare \tag{4.5}$$

5. EXISTENCE AND UNIQUENESS FOR THE COMBUSTION VARIABLES

5.1. Statement of the problem and main results

The aim of this section is to study the subsystem of the reaction-diffusion equations for the temperature and mass fraction:

$$\left\{\begin{array}{l} \left\{\begin{array}{l} \Theta_t - \Theta_{xx} = \Omega(Y,\Theta) = Y^n f(\Theta)\,, \\ Y_t - \dfrac{Y_{xx}}{Le} = -\,\Omega(Y,\Theta)\,, \qquad \text{for } x \in \mathbb{R}\,,\ t \in \mathbb{R}_+\,; \end{array}\right. \\ \Theta(x,0) = \Theta_0(x)\,,\ Y(x,0) = Y_0(x)\,, \end{array}\right. \tag{5.1}$$

$$\left\{\begin{array}{l} \Theta(-\infty,t) = 0\,,\ \Theta(+\infty,t) = 1\,, \\ Y(-\infty,t) = 1\,,\ Y(+\infty,t) = 0\,; \end{array}\right. \tag{5.2}$$

Before stating the results concerning the existence and uniqueness of a solution of problem (5.1)-(5.2) we introduce a new formulation of this problem. In order to apply some of the results recalled in the preceding section, we define new unknowns (φ,ψ) satisfying zero boundary condition:. we therefore use the functions γ and γ_1 introduced in (3.7), define (φ_0,ψ_0) as in (3.8) and set:

$$\left\{\begin{array}{l} \varphi(x,t) = \Theta(x,t) - \gamma(x)\,, \\ \psi(x,t) = Y(x,t) - \gamma_1(x)\,; \end{array}\right. \tag{5.3}$$

Finally we extend the domain of definition of f by setting: $f \equiv 0$ on $\mathbb{R}_-$, and we define g by:

$$g(\xi) = \left\{\begin{array}{ll} 0 & \text{if } \xi \le 0\,, \\ \xi^n & \text{if } \xi \geqq 0\,. \end{array}\right. \tag{5.4}$$

The system (5.1)-(5.2) can now be rewritten as:

$$\left\{\begin{array}{l} \varphi_t - \varphi_{xx} = f(\varphi + \gamma)\, g(\psi + \gamma_1) + \gamma_{xx}\,, \\ \psi_t - \dfrac{\psi_{xx}}{Le} = -\,f(\varphi + \gamma)\, g(\psi + \gamma_1) - \dfrac{\gamma_{xx}}{Le}\,; \\ \varphi(x,0) = \varphi_0(x)\,,\ \psi(x,0) = \psi_0(x)\,; \end{array}\right. \tag{5.5}$$

$$\varphi(-\infty,t) = \varphi(+\infty,t) = \psi(-\infty,t) = \psi(+\infty,t) = 0\,. \tag{5.6}$$

The next lemma shows that problem (5.5) does belong to the general framework of the preceding section. Consider the linear operator:

$$A : \left| \begin{array}{l} D(A) = H^2 \times H^2 \to L^2 \times L^2 , \\ (\varphi,\psi) \to (-\varphi_{xx} , -\dfrac{\psi_{xx}}{Le}) . \end{array} \right.$$

We then have:

LEMMA 5.1:

A is a maximal monotone self-adjoint operator. ▪

PROOF: It is obvious from Lemma 4.5. ▪

REMARK 5.2: Let S^2 be the continuous linear semigroup generated by $-A$. The two following properties follow easily from Lemma 4.6:

$$\forall\, p \in [1,\infty) ,\ \forall\, (\varphi,\psi) \in L^2 \times L^2 \cap L^p \times L^p ,\ \forall\, t \in \mathbb{R}_+ ,\ \|S^2(t)(\varphi,\psi)\|_p \leqq \|(\varphi,\psi)\|_p ;$$

$$\forall\, (\varphi,\psi) \in L^2 \times L^2 \cap L^\infty \times L^\infty ,\ S^2(.)(\varphi,\psi) \in C(\mathbb{R}_+, L^\infty \times L^\infty) . \ ▪$$

We can now make precise what the solutions of (5.5) may be, in view of Definitions 3.2 and 4.2. For the more general problem:

$$\left\{ \begin{array}{l} \varphi_t - \varphi_{xx} = h_1(\varphi,\psi,x) , \\ \psi_t - \dfrac{\psi_{xx}}{Le} = h_2(\varphi,\psi,x) , \\ \varphi(x,0) = \varphi_0(x) ,\ \psi(x,0) = \psi_0(x) ; \end{array} \right. \tag{5.7}$$

we state:

DEFINITION 5.3:

(φ,ψ) is a *weak solution* of (5.7) on $\mathbb{R} \times [0,T)$ if and only if:

$$\begin{cases} \varphi,\psi \in C([0,T),L^2)\ ,\ H(\varphi,\psi) \in C([0,T),L^2 \times L^2)\ , \\ \forall\ t \in [0,T)\ ,\ (\varphi,\psi)(t) = S^2(t)(\varphi_0,\psi_0) + \int_0^t S^2(t-s)\,H[(\varphi,\psi)(s)]\,ds\ , \end{cases}$$

where $H(\varphi,\psi) = [h_1(\varphi,\psi,x),h_2(\varphi,\psi,x)]$.

A *weak solution* (φ,ψ) of (5.7) is a *classical solution* on the interval K of $\mathbb{R}_+$ if and only if:

A classical solution (φ,ψ) of (5.7) is a *smooth solution* if and only if (φ,ψ) and all the partial derivatives appearing in the equations (5.7) are continuous with respect to both variables x and t. ▪

DEFINITION 5.4:

(Θ,Y) is a *weak* (resp: *classical*, *smooth*) solution of (5.1) on $\mathbb{R} \times [0,T)$ if and only if (Θ,Y) is related to a *weak* (resp: *classical*, *smooth*) solution (φ,ψ) of (5.5) on $\mathbb{R} \times [0,T)$ by (5.3), and satisfies:

$$\begin{cases} \Theta \in L^\infty_{loc}(\mathbb{R}_+,L^\infty)\ ; \\ \Theta(x,t) \geq 0\ ,\ \ 0 \leq Y(x,t) \leq 1\ \text{ a.e. on } \mathbb{R} \times [0,T)\ . \end{cases}$$ ▪

REMARK 5.5: It is easily checked that a *weak solution* of (5.5) is a solution in the sense of the distributions: let (φ,ψ) be a *weak solution* of (5.5) on $\mathbb{R} \times \mathbb{R}_+$, and let $\eta \in D(\mathbb{R}\times\mathbb{R}_+)$. Assuming that $\mathrm{Supp}(\eta) \subset (-M,M) \times [0,T)$, we set: $K = (-M,M) \times [0,T)$ and $K_\varepsilon = (-M,M) \times (\varepsilon,T)$. Since (φ,ψ) is a classical solution on $\mathbb{R} \times K_\varepsilon$, φ, ψ and η are in $H^1(K_\varepsilon)$. We can then apply Green's formula to get:

$$\int_{K_\varepsilon} [\, -\varphi\eta_t - (\varphi + \gamma)\eta_{xx} - f\,(\varphi + \gamma)\ g\,(\psi + \gamma_1)\eta\ ] \ =\ \int_{-M}^{M} \varphi(x,\varepsilon)\eta(x,\varepsilon)\ dx\ \ .$$

As $\varphi \in C([0,T),L^2)$, we can take the limit $\varepsilon \to 0$ in the last relation to get:

$$\int_K [\, -\varphi\eta_t - (\varphi + \gamma)\eta_{xx} - f\,(\varphi + \gamma)\ g\,(\psi + \gamma_1)\eta\ ] \ =\ \int_{-M}^{M} \varphi_0(x)\eta(x,0)\ dx\ \ ,$$

which (together with the analogous relation for ψ) shows that (φ,ψ) is a solution of (5.5) in the sense of $D'(\mathbb{R}\times\mathbb{R}_+)$.

In the same way, a *weak solution* (Θ,Y) of (5.1) is a solution in the sense of distributions. ▪

We are now ready to state the main results about problems (5.1)-(5.2) and (5.5)-(5.6). For the sake of simplicity, we are using both the new unknowns (φ,ψ) and the old ones (Θ,Y).

THEOREM 5.6:

Under the hypotheses (3.9) and (3.10), there exists a unique solution (Θ,Y) of problem (5.1)-(5.2) on $\mathbb{R}\times\mathbb{R}_+$. The corresponding solution (φ,ψ) of (5.5)-(5.6) satisfies:

$$\varphi,\psi \in C(\mathbb{R}_+,L^2 \cap L^\infty) \cap C^1(\mathbb{R}_+^*,L^2) \cap C(\mathbb{R}_+^*,H^2)$$

COROLLARY 5.7:

Under the hypotheses (3.9), (3.10) and (3.15), the solution (Θ,Y) of (5.1)-(5.2) is a *smooth solution* on $\mathbb{R}\times\mathbb{R}_+^*$.

Moreover, if $\varphi_0,\psi_0 \in H^4$, (Θ,Y) is a *smooth solution* on $\mathbb{R}\times\mathbb{R}_+$. ▪

5.2. A lemma for systems of type (5.5)

We begin the proof of the above theorems with the next result, which will be used several times in the sequel:

LEMMA 5.8:

Let f_0 and g_0 be two bounded Lipschitz-continuous functions on $\mathbb{R}$, with $f_0(0) = 0$, $g_0(0) = 0$; consider the problem:

$$\begin{cases} \varphi_t - \varphi_{xx} = f_0(\varphi+\gamma)\, g_0(\psi+\gamma_1) + \gamma_{xx}\,, \\ \psi_t - \dfrac{\psi_{xx}}{Le} = -f_0(\varphi+\gamma)\, g_0(\psi+\gamma_1) - \dfrac{\gamma_{xx}}{Le}\,; \\ \varphi(x,0) = \varphi_0(x)\,,\ \psi(x,0) = \psi_0(x)\,. \end{cases} \tag{5.8}$$

For any $(\varphi_0,\psi_0) \in L^2 \times L^2$, the problem (5.8) has a unique solution (φ,ψ) in $C(\mathbb{R}_+,L^2 \times L^2)$; this solution is a *classical solution* on $\mathbb{R}_+^\bullet$:

$$\varphi,\psi \in C(\mathbb{R}_+^\bullet,H^2) \cap C^1(\mathbb{R}_+^\bullet,H^2) . \blacksquare$$

PROOF: Define the mapping F_0 by:

$$F_0(\varphi,\psi) = [\, f_0(\varphi+\gamma)\, g_0(\psi+\gamma_1) + \gamma_{xx} \,,\; - f_0(\varphi+\gamma)\, g_0(\psi+\gamma_1) - \frac{\gamma_{xx}}{Le} \,]$$

for $\varphi,\psi \in L^2$. In view of Theorem 4.3, it suffices to show that F_0 is a Lipschitz-continuous mapping from $L^2 \times L^2$ into itself.

Let $h = f_0(\varphi+\gamma)\, g_0(\psi+\gamma_1)$. It is classical to show that $h \in L^2$ when $\varphi,\psi \in L^2$. Let us simply check that h is Lipschitz-continuous from $L^2 \times L^2$ into L^2 . Let M_f , M_g , L_f , L_g be real constants such that:

$$\forall\, \xi \in \mathbb{R} ,\quad |f_0(\xi)| \le M_f \,,\quad |g_0(\xi)| \le M_g \,;$$

$$\forall\, (\xi,\eta) \in \mathbb{R}^2 ,\quad |f_0(\xi) - f_0(\eta)| \le L_f\, |\xi - \eta| \,,\quad |g_0(\xi) - g_0(\eta)| \le L_g\, |\xi - \eta| \,.$$

For $\varphi_1,\psi_1 \in L^2$, $\varphi_2,\psi_2 \in L^2$, we have:

$$\begin{aligned} h_1 - h_2 &= f_0(\varphi_1+\gamma)\, g_0(\psi_1+\gamma_1) - f_0(\varphi_2+\gamma)\, g_0(\psi_2+\gamma_1) \\ &= f_0(\varphi_1+\gamma)\, [g_0(\psi_1+\gamma_1) - g_0(\psi_2+\gamma_1)] + g_0(\psi_2+\gamma_1)\, [f_0(\varphi_1+\gamma) - f_0(\varphi_2+\gamma)] \,; \end{aligned}$$

whence:

$$\begin{aligned} \|h_1 - h_2\|_2 &\le M_f\, L_g\, \|\psi_1 - \psi_2\|_2 + M_g\, L_f\, \|\varphi_1 - \varphi_2\|_2 \,, \\ &\le [M_f L_g + M_g L_f]\, \|(\varphi_1 - \varphi_2 \,,\, \psi_1 - \psi_2)\|_2 \,, \end{aligned}$$

and the proof is complete. ▪

5.3. Uniqueness

The uniqueness of the solution (Θ,Y) of problem (5.1)-(5.2) is a consequence of the following Proposition:

PROPOSITION 5.9:

Let $T > 0$. Under the hypotheses (3.9), there exists at most one solution of problem (5.5) in $C([0,T),L^2\times L^2) \cap L^\infty([0,T),L^\infty\times L^\infty)$. ▪

PROOF: Let $T > 0$, and let (φ_1,ψ_1) and (φ_2,ψ_2) be two solutions of (5.5), with $\varphi_i,\psi_i \in L^\infty([0,T),L^\infty)$ for $i = 1,2$. Choosing $U \in \mathbb{R}$ such that $\|(\varphi_i,\psi_i)(t)\|_\infty \leq U$ for $i = 1,2$ and $t \in [0,T)$, we can consider two functions f_U and g_U satisfying:

$$\begin{cases} f_U \text{ is positive, bounded and Lipschitz-continuous on } \mathbb{R}\,, \\ f_U(\xi) = f(\xi) \ \text{ if } |\xi| \leq U\,. \end{cases} \tag{5.9}$$

$$\begin{cases} g_U \text{ is positive, bounded and Lipschitz-continuous on } \mathbb{R}\,, \\ g_U(\xi) = g(\xi) \ \text{ if } |\xi| \leq U\,. \end{cases} \tag{5.10}$$

(φ_1,ψ_1) and (φ_2,ψ_2) are then solutions of the following problem:

$$\begin{cases} \varphi_t - \varphi_{xx} = f_U(\varphi + \gamma)\, g_U(\psi + \gamma_1) + \gamma_{xx}\,, \\ \psi_t - \dfrac{\psi_{xx}}{Le} = -\, f_U(\varphi + \gamma)\, g_U(\psi + \gamma_1) - \dfrac{\gamma_{xx}}{Le}\,; \\ \varphi(x,0) = \varphi_0(x)\,,\ \psi(x,0) = \psi_0(x)\,. \end{cases} \tag{5.11}$$

Applying Lemma 5.8, we get $(\varphi_1,\psi_1) = (\varphi_2,\psi_2)$, which ends the proof. ▪

5.4. Global existence

We show in this section the local existence of a solution (Θ,Y) of problem (5.1)-(5.2):

PROPOSITION 5.10:

Assume that the hypotheses (3.9) hold. Then there exists $T_{max} \in \mathbb{R}_+^* \cup \{+\infty\}$ such that a solution (Θ,Y) of problem (5.1)-(5.2) exists on $\mathbb{R} \times [0,T_{max})$. Moreover, (Θ,Y) is a *classical solution* on $(0,T_{max})$, and the following alternative holds:

$$\begin{cases} \text{Either:} \quad T_{max} = +\infty\,, \\ \text{Or:} \quad \lim\limits_{t\to T_{max}} \|\Theta(t)\|_\infty = +\infty\,. \ ▪ \end{cases} \tag{5.12}$$

The proof of this proposition is divided into two lemmas:

LEMMA 5.11:

Under the hypotheses (3.9), there exists $T_{max} \in \mathbb{R}_+^* \cup \{+\infty\}$ such that a solution (φ,ψ) of problem (5.5) exists on $\mathbb{R} \times [0,T_{max})$. Moreover, the following properties hold:

$$\varphi,\psi \in C([0,T_{max}),L^2) \cap C((0,T_{max}),H^2) \cap C^1((0,T_{max}),L^2)\,; \tag{5.13}$$

$$\forall\, T < T_{max}\,,\ \varphi,\psi \in L^\infty([0,T),L^\infty)\,; \tag{5.14}$$

$$\begin{cases} \text{Either:} \quad T_{max} = +\infty\,, \\ \text{Or:} \quad \lim\limits_{t \to T_{max}} \|(\varphi,\psi)(t)\|_\infty = +\infty\,. \blacksquare \end{cases} \tag{5.15}$$

PROOF: a) Let us first show the existence of a solution on $\mathbb{R} \times [0,T)$ for small positive T. For $U \geq \|(\varphi_0,\psi_0)\|_\infty + 2$, we define f_U and g_U as in (5.9)-(5.10) above and consider again the problem (5.11). Lemma 5.8 applies again and gives a solution (φ_U,ψ_U). Denoting:

$$F_U(\varphi_U,\psi_U) = [\, f_U(\varphi_U + \gamma)\, g_U(\psi_U + \gamma_1) + \gamma_{xx}\,,\ - f_U(\varphi_U + \gamma)\, g_U(\psi_U + \gamma_1) - \frac{\gamma_{xx}}{Le}\,]\,,$$

and using Remark 5.2, we get:

$$(\varphi_U,\psi_U)(t) = S^2(t)(\varphi_0,\psi_0) + \int_0^t S^2(t-s)\, F_U[(\varphi_U,\psi_U)(s)]\, ds\,, \tag{5.16}$$

$$\|(\varphi_U,\psi_U)(t)\|_\infty \leq \|(\varphi_0,\psi_0)\|_\infty + \int_0^t \|F_U[(\varphi_U,\psi_U)(s)]\|_\infty\, ds\,.$$

Since f_U, g_U and γ_{xx} are bounded, we can obviously find a constant C_U such that: $\forall\, (\varphi_1,\psi_1) \in L^\infty \times L^\infty$, $\|F_U(\varphi_1,\psi_1)\|_\infty \leq C_U$. This implies:

$$\|(\varphi_U,\psi_U)(t)\|_\infty \leq \|(\varphi_0,\psi_0)\|_\infty + C_U\, t\,.$$

Let $t_U = \dfrac{1}{C_U}$. For $t \in [0,t_U)$, we have:

$$\|(\varphi_U,\psi_U)(t)\|_\infty \leq \|(\varphi_0,\psi_0)\|_\infty + 1\,,$$

whence:

$$\|\varphi_U(t) + \gamma\|_\infty \leq \|(\varphi_0,\psi_0)\|_\infty + 2 \leq U ,$$

$$\|\psi_U(t) + \gamma_1\|_\infty \leq \|(\varphi_0,\psi_0)\|_\infty + 2 \leq U .$$

This implies that (φ_U,ψ_U) is a solution of (5.5) on $\mathbb{R} \times [0,t_U)$; this solution satisfies: $\varphi_U,\psi_U \in C([0,t_U),L^2) \cap L^\infty([0,t_U),L^\infty)$, and $\varphi_U,\psi_U \in C((0,t_U),H^2) \cap C^1((0,t_U),L^2)$.

b) Since a solution of problem (5.5) exists locally in the neighbourhood $[0,t_U)$ of 0, it is classical to show the existence of a solution (φ,ψ) satisfying (5.13),(5.14) and (5.15) on a maximal interval $[0,T_{\max})$. For sake of completeness we briefly recall the proof of (5.15): let us assume that $T_{\max} < +\infty$ and that there exists a sequence $(t_m)_{m\in\mathbb{N}}$ such that:

$$\begin{cases} \lim\limits_{m\to\infty} t_m = T_{\max} , \\ \exists\, V > 0 ,\ \forall\, m \in \mathbb{N} ,\ \|(\varphi,\psi)(t_m)\|_\infty \leq V . \end{cases} \tag{5.17}$$

Let $U = V + 2$. For $m \in \mathbb{N}$, we can argue as in a) above to show the existence of a solution of (5.5) on the interval $[t_m , t_m + t_U)$. Since t_U does not depend on m we can choose the latter so that: $t_m + t_U > T_{\max}$, which contradicts the assumption that $[0,T_{\max})$ is a maximal interval for the existence of a solution of (5.5). (5.17) is therefore wrong and the alternative (5.15) holds. ▪

The solution (φ,ψ) of (5.5) defined in Lemma 5.11 satisfies the boundary conditions (5.6) on $(0,T_{\max})$. For $t \in (0,T_{\max})$, we have indeed $\varphi(\pm\infty,t) = \psi(\pm\infty,t) = 0$ since $\varphi,\psi \in H^1$ (see [3]).

We can now end the proof of Proposition 5.10 by using the maximum principle for parabolic partial differential equations:

LEMMA 5.12:

Let (φ,ψ) be the solution of (5.5) defined in Lemma 5.11. For $(x,t) \in \mathbb{R} \times [0,T_{\max})$, define:

$$\begin{cases} \Theta(x,t) = \varphi(x,t) + \gamma(x) , \\ Y(x,t) = \psi(x,t) + \gamma_1(x) . \end{cases}$$

Then the following inequalities hold:

$$\Theta(x,t) \geqq 0\,,\ \ 0 \leqq Y(x,t) \leqq 1 \ \text{ a.e. on } \mathbb{R} \times [0,T_{\max})\,. \blacksquare \tag{5.18}$$

PROOF: a) Let us first show that $Y \geqq 0$. This is essentially the maximum principle. For any function Z of $L^2_{loc}(\mathbb{R})$ we define as usual: $Z^- = \max(0,-Z)$, $Z^+ = \max(0,Z)$. For $t \in (0,T_{\max})$, it is known that $\psi^-(t) \in H^1$, $(\psi(t)+\gamma_1)^- \in H^1_{loc}(\mathbb{R})$ (see [12]). It follows easily from the properties (3.7) of γ and γ_1 that $(\psi(t)+\gamma_1)^- = Y^- \in H^1$. Since (φ,ψ) is a *classical solution* of (5.5) on $(0,T_{\max})$, we can write:

$$Y_t\, Y^- - \frac{Y_{xx}\, Y^-}{Le} = -f(\Theta)\, g(Y)\, Y^-\,.$$

But $g(Y)Y^- \equiv 0$ from (5.4); integrating by parts the last relation, we get:

$$\frac{d}{dt}\Big[\frac{1}{2}\int_{\mathbb{R}}(Y^-)^2\Big] + \frac{1}{Le}\int_{\mathbb{R}}[(Y^-)_x]^2 = 0\,,$$

whence:

$$\frac{d}{dt}\Big[\int_{\mathbb{R}}(Y^-)^2\Big] \leqq 0 \quad \text{for } t \in (0,T_{\max})\,. \tag{5.19}$$

On the other hand, it can be checked easily that the mapping $\psi \to (\psi+\gamma_1)^-$ is continuous from L^2 into itself. Thus $Y^- \in C([0,T_{\max}),L^2)$. Since $\int_{\mathbb{R}}(Y^-)^2$ is decreasing on $(0,T_{\max})$ from (5.19) and $\int_{\mathbb{R}}[Y^-(t=0)]^2 = 0$ from (3.9), we obtain:

$$Y^-(t) \equiv 0 \quad \text{for } t \in [0,T_{\max})\,,$$

or equivalently:

$$Y(t) \geqq 0 \quad \text{for } t \in [0,T_{\max})\,.$$

b) Using $(Y-1)^+$ and Θ^- instead of Y^- gives the other inequalities (5.18) as in a) above. $\blacksquare$

5.5. Regularity of the solution

Before showing that a global solution does exist (i.e. $T_{max} = +\infty$), we can investigate the smoothness of the solution (Θ,Y) defined in Proposition 5.10; this is the aim of this section.

A first result concerning the regularity of the solution is the next Lemma, which is an obvious consequence of Theorem 4.3 and Lemma 5.11:

LEMMA 5.13:

If $(\varphi_0,\psi_0) \in H^2 \times H^2$, the solution (Θ,Y) defined in Proposition 5.10 is a *classical solution* on $[0,T_{max})$. ▪

Without any further assumption on f, we also have:

LEMMA 5.14:

The solution (Θ,Y) defined in Proposition 5.10 satisfies:

$$(\Theta,Y) \in C([0,T_{max}),L^\infty). \blacksquare \tag{5.20}$$

PROOF: Since the imbedding $H^2 \subset L^\infty$ is continuous, we already have: $(\Theta,Y) \in C((0,T_{max}),L^\infty)$ from (5.13). Therefore we only have to show that:

$$\|(\varphi,\psi)(t) - (\varphi_0,\psi_0)\|_\infty \to 0 \text{ when } t \to 0 . \tag{5.21}$$

We use again the notations of the proof of Lemma 5.11. Let $U > \|(\varphi_0,\psi_0)\|_\infty + 2$. For $t > 0$ small enough, (φ,ψ) is a solution of (5.11) and (5.16) implies:

$$\|(\varphi,\psi)(t) - (\varphi_0,\psi_0)\|_\infty \leqq \|S^2(\varphi_0,\psi_0) - (\varphi_0,\psi_0)\|_\infty + C_U\, t .$$

(5.21) follows now immediately from Remark 5.2. ▪

The next proposition shows that, with the additional assumptions (3.15) on f, there exists a *smooth solution* of (5.1)-(5.2):

PROPOSITION 5.15:

Under the hypotheses (3.15) on f, the solution (Θ,Y) of problem (5.1)-(5.2) defined in Proposition 5.10 is a *smooth solution* on $\mathbb{R}\times\mathbb{R}_+^*$. The corresponding solution (φ,ψ) of (5.5)-(5.6) satisfies:

$$\varphi,\psi \in C((0,T_{\max}),H^4) \cap C^1((0,T_{\max}),H^2) \cap C^2((0,T_{\max}),L^2)\,. \blacksquare$$

REMARK 5.16: This regularity result holds without any assumption on the regularity of the initial data (φ_0,ψ_0) - only (3.9) is assumed. This is of course related to the strong regularizing effect of the heat equation. ▪

COROLLARY 5.17:

Assume that the hypotheses (3.15) hold, and that $\varphi_0,\psi_0 \in H^4$. Then the solution (Θ,Y) of problem (5.1)-(5.2) defined in Proposition 5.10 is a *smooth solution* on $\mathbb{R}\times\mathbb{R}_+$. The corresponding solution (φ,ψ) of (5.5)-(5.6) satisfies:

$$\varphi,\psi \in C([0,T_{\max}),H^4) \cap C^1([0,T_{\max}),H^2) \cap C^2([0,T_{\max}),H^2)\,. \blacksquare$$

We begin the proof of Proposition 5.15 with two lemmas. Assuming (3.15), we first introduce two functions $\hat{f}$ and $\hat{g}$ satisfying:

$$\begin{cases} \hat{f} \in C^2(\mathbb{R},\mathbb{R})\,, \\ \forall\ \xi > 0\,,\ \hat{f}_{xx} \text{ is Lipschitz-continuous on } [-\xi,\xi]\,, \\ \forall\ \xi > 0\,,\ \hat{f}(\xi) = f(\xi)\,; \end{cases}$$

$$\hat{g}(\xi) = \xi^n\,;$$

and a mapping $\hat{F}$ defined by:

$$\hat{F}(\varphi,\psi) = [\ \hat{f}(\varphi+\gamma)\,\hat{g}(\psi+\gamma_1) + \gamma_{xx}\,,\ -\hat{f}(\varphi+\gamma)\,\hat{g}(\psi+\gamma_1) - \frac{\gamma_{xx}}{Le}\ ]\,,$$

for $\varphi,\psi \in L^2$.

LEMMA 5.18:

Under the hypotheses (3.9) and (3.15), the mapping $\hat{F}$ is Lipschitz-continuous from any bounded subset of $H^2{\times}H^2$ into $H^2{\times}H^2$. ▪

PROOF: a) Let us first show that $\widehat{F}(\varphi,\psi) \in H^2 \times H^2$ when $\varphi,\psi \in H^2$. For $\varphi,\psi \in H^2$, let $\widehat{h} = \widehat{f}(\varphi+\gamma)\,\widehat{g}(\psi+\gamma_1)$, $M = \|(\varphi,\psi)\|_\infty$. We define: $M_f = \max_{[-M,M]} \widehat{f}$, $M_g = \max_{[-M,M]} \widehat{g}$, $L_f = \max_{[-M,M]} \widehat{f}_x$, $L_g = \max_{[-M,M]} \widehat{g}_x$. Thus $\widehat{h} \in L^2$ as in the proof of Lemma 5.8. Furthermore, we have:

$$\widehat{h}_x = \widehat{f}_x(\varphi+\gamma)(\varphi_x+\gamma_x)\widehat{g}(\psi+\gamma_1) + \widehat{f}(\varphi+\gamma)\widehat{g}_x(\psi+\gamma_1)(\psi_x+\gamma_{1_x}) ,$$

$$|\widehat{h}_x| \leqq L_f\, M_g\, |\varphi_x+\gamma_x| + M_f\, L_g\, |\psi_x+\gamma_{1_x}| ,$$

which yields that $\widehat{h}_x \in L^2$. It can also be shown that $\widehat{h}_{xx} \in L^2$, using the Sobolev continuous imbedding:

$$\begin{cases} H^2 \subset W^{1,\infty}(\mathbb{R}) , \\ \exists\, S>0 ,\ \forall\, \varphi \in H^2 ,\ \|\varphi\|_{W^{1,\infty}} \leqq S\, \|\varphi\|_{H^2} . \end{cases}$$

b) It is long but easy to check that, for any $M>0$, $\widehat{h}$ is Lipschitz-continuous from $\{ (\varphi,\psi) \in H^2 \times H^2 ,\ \|(\varphi,\psi)\|_{H^2\times H^2} \leqq M \}$ into H^2 ; the details are left to the reader. ▪

For $\varphi_1,\psi_1 \in L^2$, we now consider the problem:

$$\begin{cases} \varphi_t - \varphi_{xx} = \widehat{f}(\varphi+\gamma)\,\widehat{g}(\psi+\gamma_1) + \gamma_{xx} , \\ \psi_t - \dfrac{\psi_{xx}}{Le} = -\,\widehat{f}(\varphi+\gamma)\,\widehat{g}(\psi+\gamma_1) - \dfrac{\gamma_{xx}}{Le} ; \\ \varphi(x,0) = \varphi_1(x) ,\ \psi(x,0) = \psi_1(x) . \end{cases} \tag{5.22}$$

LEMMA 5.19:

Assume that (3.9) and (3.15) hold. Then there exists $\widehat{T}_{\max} \in \mathbb{R}_+^* \cup \{+\infty\}$ such that a unique solution $(\widehat{\varphi},\widehat{\psi})$ of problem (5.22) exists on $\mathbb{R} \times [0,\widehat{T}_{\max})$. This solution satisfies:

$$\widehat{\varphi},\widehat{\psi} \in C((0,\widehat{T}_{\max}),H^4) \cap C^1((0,\widehat{T}_{\max}),H^2) ,$$

and the following alternative holds:

$$\begin{cases} \text{Either:} \quad \widehat{T}_{\max} = +\infty , \\ \text{Or:} \quad \lim\limits_{t \to T_{\max}} \|(\widehat{\varphi},\widehat{\psi})(t)\|_{H^2\times H^2} = +\infty . \ ▪ \end{cases}$$

PROOF: From Lemma 5.18, it suffices to apply Theorem 4.4 with $H = H^2 \times H^2$, $D(A) = H^4 \times H^4$ and $F = \widehat{F}$. ▪

We can now complete the proof of Proposition 5.15 and Corollary 5.17:

PROOF: a) Let (φ,ψ) be the solution of (5.5) defined in Lemma 5.11; for $\varepsilon \in (0,T_{max})$, we set $(\varphi_1,\psi_1) = (\varphi,\psi)(t=\varepsilon) \in H^2 \times H^2$. Applying Lemma 5.19, we get a solution $(\widehat{\varphi},\widehat{\psi})$ of (5.22), which is unique in $C((0,\widehat{T}_{max}),H^2 \times H^2)$. But it is straightforward to see that $(\varphi,\psi)(t+\varepsilon)$ is also a solution of (5.22) in $C([0,T_{max}-\varepsilon),H^2 \times H^2)$. These two solutions coincide, and we get:

$$(\varphi,\psi)(t) = (\widehat{\varphi},\widehat{\psi})(t-\varepsilon) \quad \text{for } t \in [\varepsilon,T_{max}) .$$

b) Lemma 5.19 and a) above obviously imply that the solution (φ,ψ) defined in Lemma 5.11 satisfies:

$$\varphi,\psi \in C((0,T_{max}),H^4) \cap C^1((0,T_{max}),H^2) .$$

It suffices now to use the Sobolev continuous imbeddings $H^2 \subset C^1(\mathbb{R})$, $H^4 \subset C_3(\mathbb{R})$ to show that (φ,ψ) is a smooth solution of (5.5)-(5.6) on $(0,T_{max})$. To end the proof of Proposition 5.15, it remains to show that $\varphi,\psi \in C^2((0,T_{max}),L^2)$, or equivalently that $\widehat{F}(\varphi,\psi) \in C^1((0,T_{max}),L^2 \times L^2)$; this is straightforward and is left to the reader. ▪

5.6. Existence for all time

We now end this fifth section by showing that $T_{max} = +\infty$:

PROPOSITION 5.20:

Under the hypothesis (3.10), the solution (Θ,Y) of (5.1)-(5.2) defined in Proposition 5.10 exists on $\mathbb{R} \times \mathbb{R}_+$:

$$T_{max} = +\infty . \ ▪ \tag{5.23}$$

PROOF: For $p \in [1,+\infty)$ and $t \in (0,T_{max})$, we can write, since (Θ,Y) is a *classical solution* on $(0,T_{max})$:

$$\Theta_t \, \Theta^{p-1} - \Theta_{xx} \, \Theta^{p-1} = Y^n \, f(\Theta) \, \Theta^{p-1} .$$

Integrating by parts as in the proof of Lemma 5.12, we obtain:

$$\frac{1}{p} \frac{d}{dt} \Big(\int_{\mathbb{R}} \Theta^p \Big) \leq \int_{\mathbb{R}} [\, Y^n f(\Theta) \, \Theta^{p-1} \,] .$$

(3.10) and (5.18) now imply:

$$\frac{d}{dt} \Big(\int_{\mathbb{R}} \Theta^p \Big) \leq C_f \, p \int_{\mathbb{R}} \Theta^p .$$

Let $t_0 \in (0, T_{max})$. Applying Gronwall's Lemma to the last inequality, we can write:

$$\int_{\mathbb{R}} \Theta(t)^p \leq \int_{\mathbb{R}} \Theta(t_0)^p \, e^{pC_f(t-t_0)} ,$$

or:

$$\|\Theta(t)\|_p \leq \|\Theta(t_0)\|_p \, e^{C_f(t-t_0)} .$$

We can then take the limit $p \to \infty$ to get:

$$\|\Theta(t)\|_\infty \leq \|\Theta(t_0)\|_\infty \, e^{C_f(t-t_0)} ,$$

which together with (5.12) implies $T_{max} = +\infty$. ▪

Of course, from a physical standpoint, it can be thought that (5.23) holds even if (3.10) is not assumed, because of (5.12), since one may expect that the increase of the temperature is limited by the consumption of the reactant. Nevertheless, we have been able to prove rigorously the global existence of the solution only with the assumption (3.10), or in the following case:

LEMMA 5.21:

Assume that the hypotheses (3.9) hold. If moreover $Le = 1$, then the solution (Θ, Y) of (5.1)-(5.2) exists on $\mathbb{R} \times \mathbb{R}_+$. ▪

PROOF: If $Le = 1$, we can add the two equations (5.1) to get:

$$(Y + \Theta)_t - (Y + \Theta)_{xx} = 0 .$$

A straightforward application of the maximum principle for parabolic partial differential equations yields:

$$\|(Y + \Theta)(t)\|_\infty \leq \|(Y + \Theta)(0)\|_\infty ,$$

and (5.23) follows again from (5.12). ▪

REMARK 5.22: With the same hypothesis $Le = 1$, it can be shown that $(Y + \Theta)$ converges towards 1 uniformly on $\mathbb{R}$ as t tends to $+\infty$:

$$\lim_{t \to +\infty} \|Y + \Theta - 1\|_\infty = 0 . \blacksquare$$

6. EXISTENCE AND UNIQUENESS FOR THE HYDRODYNAMICAL VARIABLES

6.1. Statement of the problem and main results

We now want to consider the subsystem (2.11) for the hydrodynamical variables - density, velocity and pressure:

$$(\Theta + \alpha)\,\rho = 1\,; \tag{6.1}$$

$$\begin{cases} u_x = \Theta_t\,, \\ u(-\infty,t) = u^0\,; \end{cases} \tag{6.2}$$

$$\begin{cases} u_t + p_x = 0\,, \\ p(-\infty,t) = 0\,; \end{cases} \tag{6.3}$$

Throughout this section, it will be assumed that the hypotheses (3.9)-(3.10) hold. The solution (Θ,Y) of (5.1)-(5.2) in $\mathbb{R} \times \mathbb{R}_+$ and the corresponding solution (φ,ψ) of (5.5)-(5.6) are now considered given. We let $\Omega(y,t) = \Omega[Y(y,t),\Theta(y,t)]$.

We recall that the *weak* or *smooth solutions* of (6.1)-(6.3) are defined at the beginning of Section 3 (in particular, the boundary condition (6.3.b) is fulfilled in the sense of (3.6) for *weak solutions*). About problem (6.1)-(6.3), we are going to prove:

THEOREM 6.1:

Assume that the hypotheses (3.9)-(3.13) hold. Then there exists a unique *weak solution* (ρ,u,p) of (6.1)-(6.3) in $\mathbb{R} \times \mathbb{R}_+$.

If moreover (3.14) and (3.15) hold, (ρ,u,p) is a *smooth solution* on $\mathbb{R} \times \mathbb{R}_+$. ∎

In order to prove this result, we now solve the two problems (6.2) and (6.3) in sequence [solving (6.1) for the density ρ is an obvious task since $\Theta(x,t) + \alpha \geqq \alpha > 0$ for all $(x,t) \in \mathbb{R} \times \mathbb{R}_+$].

6.2. Velocity

PROPOSITION 6.2:

There exists a unique *weak solution* of (6.2) in $\mathbb{R} \times \mathbb{R}_+$:

$$u(x,t) = u^0 + \Theta_x(x,t) + \int_{-\infty}^{x} \Omega(y,t)\, dy \quad \text{for } (x,t) \in \mathbb{R} \times \mathbb{R}_+ . \tag{6.4}$$

and u is a *smooth solution* of (6.2) in $\mathbb{R} \times \mathbb{R}_+^{\bullet}$.

Moreover, if $\varphi_0, \psi_0 \in H^2$, u is a *smooth solution* of (6.2) in $\mathbb{R} \times \mathbb{R}_+$. ▪

PROOF: a) Let $t > 0$. Since (Θ, Y) is a *classical solution* of (5.1) in the neighbourhood of t, we have from (6.2):

$$u_x = \Theta_t = \Theta_{xx} + \Omega \quad \in L^1_{loc}(\mathbb{R}),$$

whence:

$$u(x,t) = u(0,t) + \Theta_x(x,t) - \Theta_x(0,t) + \int_0^x \Omega(y,t)\, dy .$$

Since we want a finite limit $u(-\infty,t)$ to exist, we only need to show that:

$$\int_{-\infty}^{0} \Omega(y,t)\, dy < +\infty. \tag{6.5}$$

But (3.10) and (5.18) imply: $\Omega(y,t) \leq C_f\ \Theta(y,t)$ and (6.5) follows from the relation $\Theta(t) \in H^2 \subset L^1$. We then obtain (6.4) for $t > 0$ [we have $\Theta_x(-\infty,t) = 0$ since $\Theta(t) \in H^2$].

b) It is clear that the solution u defined by (6.4) for $t > 0$ satisfies (6.2.a) in the sense of $D'(\mathbb{R}\times\mathbb{R}_+)$. When $\varphi_0, \psi_0 \in H^2$, we can argue as in a) above for $t = 0$ and obtain (6.4) for $t \geq 0$. To show that u is then a *smooth solution* of (6.2), it remains to prove that $\int_{-\infty}^{x} \Omega(y,t)\, dy$ is continuous with respect to both variables x and t; this will be a consequence of the next lemma. ▪

Before studying the pressure problem (6.3), we state some results about the regularity of the velocity u.

LEMMA 6.3:

The solution u of (6.2) satisfies:

$$u \in C(\mathbb{R}_+^*, C(\mathbb{R})) \cap C(\mathbb{R}_+^*, L^\infty)\ .$$

If moreover $\varphi_0, \psi_0 \in H^2$, then $u \in C(\mathbb{R}_+, C(\mathbb{R})) \cap C(\mathbb{R}_+, L^\infty)$. ▪

PROOF: For $T > 0$ define $M = \max_{t \in [0,T]} \|\Theta(t)\|_\infty$. For $t \in (0,T]$, we first have $u(t) \in L^\infty$, or equivallently $\Omega(t) \in L^1$ from the estimates:

$$\begin{cases} \Omega(t) \le C_f\, \Theta(t) \in C(\mathbb{R}_+^*, L^1(\mathbb{R}_-^*))\ , \\ \Omega(t) \le C_f\, MY(t) \in C(\mathbb{R}_+^*, L^1(\mathbb{R}_+^*))\ . \end{cases}$$

These two inequalities can be written together in the form $\Omega(t) \le G(t)$ with $G(t) \in C(\mathbb{R}_+^*, L^1)$. The continuity of the integral $\int \Omega(y,t)\, dy$ with respect to the variable t is now a consequence of classical convergence results from integration theory. For sake of completeness we sketch the arguments: arguing by contradiction, we assume that there exists a sequence (t_n) satisfying $t_n \to t_0 > 0$ and:

$$\|\Omega(t_n) - \Omega(t_0)\|_1 \ge \varepsilon > 0\ . \tag{6.6}$$

Then from the converse of Lebesgue's bounded convergence theorem (see [3], p. 58), there exists $G_0 \in L^1$ and a subsequence (t_{n_k}) such that $G(y, t_{n_k}) \le G_0(y)$ a.e. for all n_k. Since (5.20) proves that $\Omega(t_{n_k})$ converges pointwise towards $\Omega(t_0)$, Lebesgue's bounded convergence theorem now shows that $\Omega(t_{n_k})$ converges to $\Omega(t_0)$ in L^1 , which contradicts (6.6) and ends the proof. ▪

LEMMA 6.4:

Assume that the hypotheses (3.12) hold and define $v(x,t) = \int_{-\infty}^{x} \Omega(y,t)\, dy$.

Then:

$$v \in C^1(\mathbb{R} \times \mathbb{R}_+^*, \mathbb{R})\ , \text{ and } v_t(x,t) = \int_{-\infty}^{x} \Omega_t(y,t)\, dy\ . ▪$$

PROOF: The assumptions (3.12) obviously implies:

$$\forall M > 0 \,,\; \exists K_M > 0 \,,\; \forall \vartheta \in [0,M] \,,\; |f'(\vartheta)| \leq K_M \, \vartheta^\beta \tag{6.7}$$

(with $\beta > \frac{1}{2}$). Let again $T > 0$ and $M = \max_{t\in[0,T]} \|\Theta(t)\|_\infty$. For $t \in [0,T]$ and $y \in \mathbb{R}$ we have:

$$\Omega_t = nY^{n-1}Y_t f(\Theta) + Y^n \Theta_t f'(\Theta) \,,$$

whence: $|\Omega_t(t)| \leq nC_f \Theta(t)|Y_t(t)| + K_M \Theta^\beta(t)|\Theta_t(t)|$,

$$|\Omega_t(t)| \leq K\,[\Theta^2 + Y_t^2 + \Theta^{2\beta} + \Theta_t^2\,](t) \;\in C(\mathbb{R}_+^\bullet, L^1(\mathbb{R}_-^\bullet)) \,,$$

where K is a positive constant. The proof is then completed in the way similar to that of the previous lemma. ▪

The next result is now an obvious consequence of the above lemmas and of Proposition 5.15.

LEMMA 6.5:

Under the hypotheses (3.12) and (3.15), the solution u of (6.2) satisfies:

$$u \in C(\mathbb{R}_+^\bullet, C^2(\mathbb{R})) \cap C^1(\mathbb{R}_+^\bullet, C(\mathbb{R})) \,.$$

If moreover $\varphi_0, \psi_0 \in H^4$, then $u \in C(\mathbb{R}_+, C^2(\mathbb{R})) \cap C^1(\mathbb{R}_+, C(\mathbb{R}))$. ▪

6.3. Pressure

We now investigate the problem (6.3) for the pressure. We are going to prove:

PROPOSITION 6.6:

Assume that the hypotheses (3.11)-(3.13) hold and that $\varphi_0, \psi_0 \in H^2$. Then there exists a unique *weak solution* of (6.3) on $\mathbb{R} \times \mathbb{R}_+$.

If moreover the hypotheses (3.14) and (3.15) hold, there exists a unique *smooth solution* of (6.3) on $\mathbb{R} \times \mathbb{R}_+$. ▪

PROOF: a) If p is a solution of (6.3), we get from (6.4) and Lemma 6.4:

$$p_x = -\Theta_{xt} - \int_{-\infty}^{x} \Omega_t \,,$$

whence:

$$p = -\Theta_t - \int_{-\infty}^{x} [dy \int_{-\infty}^{y} \Omega_t] \,,$$

$$\text{or}\quad p(x,t) = -\Theta_{xx}(x,t) - \Omega(x,t) - \int_{-\infty}^{x} [dy \int_{-\infty}^{y} \Omega_t(z,t)\, dz] \,. \tag{6.8}$$

because of the boundary condition (6.3.b).

Therefore, we need to prove that the last integral does exist when the assumptions (3.11)-(3.13) hold. This amounts to showing that $\Omega_t(y,t)$ vanishes at $-\infty$ at least as fast as some negative power of y. More precisely, we are going to show that:

$$\exists\, \varepsilon > 0\,,\ \forall\, y < 0\,,\ \int_{-\infty}^{y} |\Omega_t| \leq \frac{1}{|y|^{1+\varepsilon}} \,. \tag{6.9}$$

We first need to introduce the functional space:

$$W_\nu = \{\, w \in L^2 \cap L^\infty\,,\ \max_{y \in \mathbb{R}_-^*} |y^\nu w(y)| \ < \ +\infty \,\}\,,$$

for $\nu > 0$, with the norm: $\|w\|_{W_\nu} = \|w\|_2 + \|w\|_\infty + \|y^\nu w\|_{L^\infty(\mathbb{R}_-^*)}$, and to state the the next lemma, which is proved at the end of this section:

LEMMA 6.7:

Let $\nu > 0$ be given. If $\varphi_0 \in W_\nu$, then $\varphi(t) \in W_\nu$ for any $t > 0$. ▪

We now have:

$$|\Omega_t| \ = \ |Y^n f'(\Theta)\Theta_t + nY^{n-1}Y_t f(\Theta)| \ \leq \ |f'(\Theta)|\ |\Theta_t| + n\,|Y_t|\ |f(\Theta)| \,.$$

Let $T > 0$ and $M = \max_{t \in [0,T]} \|\Theta(t)\|_\infty$. Since $(\Theta_t, Y_t) \in C([0,T], L^2 \times L^2)$, we can set $M' = \max_{t \in [0,T]} \|\Theta_t(t)\|_2$. For $t \in [0,T]$ and $y < 0$ we have:

$$\int_{-\infty}^{y} |f'(\Theta)|\ |\Theta_t| \ \leq \ \left[\int_{-\infty}^{y} f'(\Theta)^2\right]^{\frac{1}{2}} \left[\int_{-\infty}^{y} \Theta_t^2\right]^{\frac{1}{2}} \,.$$

by the Cauchy-Schwarz inequality. Hence, using (6.7):

$$\int_{-\infty}^{y} |f'(\Theta)|\ |\Theta_t| \leq M' K_M \left[\int_{-\infty}^{y} \Theta^{2\beta}\right]^{\frac{1}{2}},$$

As $\varphi_0 \in W_\mu$ with $\beta\mu > \dfrac{3}{2}$ from (3.13), we can apply Lemma 6.7 to get:

$$\int_{-\infty}^{y} |f'(\Theta)|\ |\Theta_t| \leq K \left[\int_{-\infty}^{y} \frac{dz}{|z|^{2\beta\mu}}\right]^{\frac{1}{2}} \leq \frac{K}{|y|^{\beta\mu-\frac{1}{2}}}.$$

Since (6.7) implies $f(\Theta) \leq K'_M \Theta^{\beta+1}$ we can argue in the same way for $\int_{-\infty}^{y} |f(\Theta)|\ |Y_t|$ and (6.9) holds.

b) It is straightforward to check that p defined by (6.8) is a solution of (6.3.a) in the sense of $D'(\mathbb{R}\times\mathbb{R}_+)$ (it suffices to argue as in Remark 5.5 and to use Lemma 6.3 for the continuity of u in the neighbourhood of $t = 0$). (u,p) is the unique *weak solution* of (6.3) in the sense of Definition 3.1. Furthermore, if (3.14) and (3.15) hold, (6.3.a) is fulfilled in the classical sense and p is a *smooth solution* of (6.3). ▪

It remains now to prove Lemma 6.7. We begin the proof with a property of the linear semigroup $S(t)$ generated by the heat operator (see the end of Section 4):

LEMMA 6.8:

Let $\nu > 0$. The operator $S(t)$ maps W_ν into itself : for any $T > 0$, there exists a positive constant M_T such that:

$$\forall\, w_0 \in W_\nu,\ \ \forall\, t \in [0,T],\ \ \|S(t)w_0\|_{W_\nu} \leq M_T\, \|w_0\|_{W_\nu}. \ ▪$$

PROOF: Let $\nu > 0$, $w_0 \in W_\nu$, $T > 0$, $t \in [0,T]$. Lemma 4.6 implies:

$$\|S(t)w_0\|_2 + \|S(t)w_0\|_\infty \leq \|w_0\|_2 + \|w_0\|_\infty .$$

Therefore it remains to study $\|y^\nu S(t)w_0\|_{L^\infty(\mathbb{R}_-^*)}$.

Let $x < 0$; we have from (4.5):

$$|x|^\nu S(t)w_0(x) = \frac{|x|^\nu}{\sqrt{4\pi t}} \int_{-\infty}^{\frac{x}{2}} w_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy + \frac{|x|^\nu}{\sqrt{4\pi t}} \int_{\frac{x}{2}}^{+\infty} w_0(y)\, e^{-\frac{|x-y|^2}{4t}}\, dy .$$

Let us denote by $A(x)$ and $B(x)$ the two terms in the right-hand side of this relation. For $y \in (-\infty, \frac{x}{2}]$, we have:

$$|w_0(y)| \le \frac{\|w_0\|_{W_\nu}}{|y|^\nu} \le 2^\nu \frac{\|w_0\|_{W_\nu}}{|x|^\nu} .$$

Thus:

$$|A(x)| \le \frac{2^\nu}{\sqrt{4\pi t}} \|w_0\|_{W_\nu} \int_{\mathbb{R}} e^{-\frac{|x-y|^2}{4t}}\, dy = 2^\nu \|w_0\|_{W_\nu} .$$

On the other hand, we also have:

$$|B(x)| \le \frac{|x|^\nu}{\sqrt{4\pi t}} \|w_0\|_\infty \int_{\frac{x}{2}}^{+\infty} e^{-\frac{|x-y|^2}{4t}}\, dy ;$$

Setting $z = \frac{y-x}{\sqrt{4t}}$ and assuming $x < -4\sqrt{T}$, we obtain:

$$\begin{aligned} |B(x)| &\le \frac{|x|^\nu}{\sqrt{\pi}} \|w_0\|_\infty \int_{\frac{|x|}{4\sqrt{t}}}^{+\infty} e^{-z^2}\, dz , \\ &\le \frac{|x|^\nu}{\sqrt{\pi}} \|w_0\|_\infty \int_{\frac{|x|}{4\sqrt{T}}}^{+\infty} e^{-z^2}\, dz \le \frac{|x|^\nu}{\sqrt{\pi}} \|w_0\|_\infty \int_{\frac{|x|}{4\sqrt{T}}}^{+\infty} e^{-z}\, dz , \\ &\le \frac{|x|^\nu}{\sqrt{\pi}}\, e^{\frac{-|x|}{4\sqrt{T}}} \|w_0\|_\infty \le K\, \|w_0\|_\infty . \end{aligned}$$

and the proof is easily achieved. ▪

PROOF of Lemma 6.7: Let $T > 0$ and $M = \max_{t\in[0,T]} \|\Theta(t)\|_\infty$. For $t \in [0,T]$ we can write:

$$\|\Omega\|_\infty \le C_f\, \|\Theta\|_\infty \le C_f\, M ,$$

$$\|\Omega(t)\|_2^2 \le \|\Omega(t)\|^2_{L^2(-\infty,-1)} + \|\Omega(t)\|^2_{L^2(-1,+1)} + \|\Omega(t)\|^2_{L^2(1,\infty)} ,$$

$$\le C_f^2\|\varphi(t)\|_2^2 + 2\, C_f^2\, M^2 + C_f^2\, M^2\, \|\psi(t)\|_2 \le K ,$$

$$\|y^\nu\Omega(t)\|_{L^\infty(\mathbb{R}^*_-)} \le C_f\, \|y^\nu\Theta(t)\|_{L^\infty(\mathbb{R}^*_-)} .$$

whence: $\|\Omega(t)\|_{W_\nu} \le K\, (\|\varphi(t)\|_{W_\nu} + 1)$, where K is a positive constant.

From (4.4) and (5.5), we have:

$$\varphi(t) = S(t)\varphi_0 + \int_0^t S(t-s)\, [\Omega(s) + \gamma_{xx}]\, ds .$$

Applying Lemma 6.8 yields:

$$\|\varphi(t)\|_{W_\nu} \le M_T\, \|\varphi_0\|_{W_\nu} + M_t \int_0^t \|\Omega(s) + \gamma_{xx}\|_{W_\nu}\, ds ,$$

$$\le K\, [\, 1 + \int_0^t \|\varphi(s)\|_{W_\nu}\, ds \,] .$$

It suffices now to apply Gronwall's lemma and the proof is complete. ∎

7. BACK TRANSFORMATION TO THE EULERIAN VARIABLES

We now want to show that the results of the preceding sections allow to show the existence of a solution for the Eulerian system (3.4)-(3.5). Since the equivalence between *smooth solutions* of the two systems (3.1)-(3.3) and (3.4)-(3.5) follows immediately from Section 2, we only have to investigate the existence and uniqueness of a *weak solution* of (3.4)-(3.5).

7.1. Coordinate transformation

We first need to study the change of variables between the Lagrangian and the Eulerian system. This is the aim of the two next lemmas:

LEMMA 7.1:

Assume that the hypotheses (3.12)-(3.14) hold and let (Θ, Y, ρ, u) be the unique weak solution of (3.1)-(3.2). Consider the mapping:

$$T_{LE} : \left| \begin{array}{l} \mathbb{R}\times\mathbb{R}_+ \to \mathbb{R}\times\mathbb{R}_+ \,, \\ \quad (x,t) \to (\xi,\tau) \quad \text{defined as:} \end{array} \right.$$

$$\left\{ \begin{array}{l} \xi(x,t) \;=\; \int_0^t u(0,t')\, dt' \;+\; \int_0^x \frac{1}{\rho(x',t)}\, dx' \,, \\ \tau(x,t) \;=\; t \,. \end{array} \right. \tag{7.1}$$

T_{LE} is a bijection from $\mathbb{R} \times \mathbb{R}_+$ into itself. Furthermore, $T_{LE} \in C^1(\mathbb{R}\times\mathbb{R}_+^*, \mathbb{R}\times\mathbb{R}_+^*)$ and:

$$\xi_x(x,t) = \frac{1}{\rho(x,t)} \,, \quad \xi_t(x,t) = u(x,t) \,. \bullet \tag{7.2}$$

PROOF: Since $u(0,t)$ and $\dfrac{1}{\rho(x,t)}$ are continuous functions on $\mathbb{R} \times \mathbb{R}_+$, the relations (7.1) define a mapping from $\mathbb{R} \times \mathbb{R}_+$ into itself. Moreover we have $\xi_x = \dfrac{1}{\rho}$ and, for $t > 0$:

$$\xi(x,t) \;=\; \int_0^t u(0,t')\, dt' \;+\; \int_0^x [\Theta(x',t) + \alpha]\, dx' \,.$$

It is clearly possible to differentiate under the second integral sign in this expression to get:

$$\xi_t(x,t) = u(0,t) + \int_0^x \Theta_t(x',t)\,dx' = u(0,t) + \int_0^x u_x(x',t)\,dx' = u(x,t).$$

On the other hand, we can easily define $T_{EL} = T_{LE}^{-1}$ by setting $T_{EL}(\xi,\tau) = (x,t)$ with:

$$\begin{cases} x(\xi,\tau) = \displaystyle\int_{\xi_0(\tau)}^{\xi} \rho(\xi',\tau)\,d\xi', \\ t(\xi,\tau) = t. \end{cases} \tag{7.3}$$

where $\xi_0(\tau) = \int_0^\tau u(0,\tau')\,d\tau'\ [= \xi(0,\tau)]$. The end of the proof is now obvious and is omitted. ▪

We can use this lemma to define the following transformation: for any $\eta \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}_+)$ we define $\hat{\eta} \in L^\infty_{loc}(\mathbb{R}\times\mathbb{R}_+)$ by:

$$\hat{\eta}(\xi,\tau) = \eta[T_{EL}(\xi,\tau)] = \eta[x(\xi,\tau),\tau]. \tag{7.4}$$

We can then state:

LEMMA 7.2:

Assume that the hypotheses (3.12)-(3.14) hold and consider the transformation $\eta \to \hat{\eta}$ defined by (7.4). The following properties hold for $p \in [1,+\infty]$ and $t_0 \in \mathbb{R}_+$:

$$\text{If } \eta \in C(\mathbb{R}_+,L^p)\,,\ \text{then } \hat{\eta} \in C(\mathbb{R}_+,L^p)\,;$$

$$\text{If } \eta \in C(\mathbb{R}_+,C(\mathbb{R}))\,,\ \text{then } \hat{\eta} \in C(\mathbb{R}_+,C(\mathbb{R}))\,;$$

$$\text{If } \eta(\pm\infty,t_0) = \eta_0\,,\ \text{then } \hat{\eta}(\pm\infty,t_0) = \eta_0\,.$$

Moreover similar properties hold for the derivatives:

$$\text{If } \eta \in C(\mathbb{R}_+,H^2)\,,\ \text{then } \hat{\eta} \in C(\mathbb{R}_+,H^2)\,;$$

$$\text{If } \eta \in C(\mathbb{R}_+,C^2(\mathbb{R}))\,,\ \text{then } \hat{\eta} \in C(\mathbb{R}_+,C^2(\mathbb{R}))\,;$$

$$\text{If } \eta \in C^1(\mathbb{R}_+,C(\mathbb{R}))\,,\ \text{then } \hat{\eta} \in C^1(\mathbb{R}_+,C(\mathbb{R}))\,. \ ▪$$

PROOF: These properties are easy to check and their proofs are omitted. We simply indicate the expressions of the partial derivatives of η and $\hat{\eta}$ which will be useful in the sequel; (7.2) and (7.4) obviously imply:

$$\begin{cases} \hat{\eta}_\xi = \rho\eta_x \,, \\ \hat{\eta}_\tau = -\rho u\eta_x + \eta_t \,; \end{cases}$$

$$\begin{cases} \eta_x = \dfrac{1}{\hat{\rho}}\hat{\eta}_\xi \,, \\ \eta_t = \hat{\eta}_\tau + \hat{u}\hat{\eta}_\xi \,. \blacksquare \end{cases}$$

7.2. Equivalence between the Lagrangian and Eulerian formulations

We can now show the existence of a weak solution to the Eulerian system (3.4)-(3.5):

PROPOSITION 7.3:

Assume that the assumptions (3.12)-(3.16) hold. Let (Θ, Y, ρ, u, p) be the unique *weak solution* of (3.1)-(3.3) and define $(\hat{\Theta}, \hat{Y}, \hat{\rho}, \hat{u}, \hat{p})$ using (7.2). Then $(\hat{\Theta}, \hat{Y}, \hat{\rho}, \hat{u}, \hat{p})$ is a *weak solution* of (3.4)-(3.5). ▪

PROOF: We only sketch the proof by studying the temperature equation. The weak solution satisfies:

$$\int_{\mathbb{R}\times\mathbb{R}_+} [-\Theta\eta_t + \Theta_x\eta_x - \Omega\eta] = \int_{\mathbb{R}} \Theta(.,0)\eta(.,0) \,, \tag{7.5}$$

for any $\eta \in D(\mathbb{R}\times\mathbb{R}_+)$. This relation also holds for $\eta \in D^1(\mathbb{R}\times\mathbb{R}_+)$ $[\eta \in C^1(\mathbb{R}\times\mathbb{R}_+)$ with compact support], since $D(\mathbb{R}\times\mathbb{R}_+)$ is dense in $D^1(\mathbb{R}\times\mathbb{R}_+)$.

Let $\hat{\eta} \in D(\mathbb{R}\times\mathbb{R}_+)$ and let η be the unique function such that $\eta(x,t) = \hat{\eta}[T_{LE}(x,t)]$. Since $\eta \in D^1(\mathbb{R}\times\mathbb{R}_+)$, (7.5) holds. Using the change of coordinates (7.1) in (7.5) gives:

$$\int_{\mathbb{R}\times\mathbb{R}_+} [-\hat{\rho}\hat{\Theta}(\hat{\eta}_\tau + \hat{u}\hat{\eta}_\xi) + \hat{\rho}\frac{\hat{\Theta}_\xi}{\hat{\rho}}\frac{\hat{\eta}_\xi}{\hat{\rho}} - \hat{\rho}\hat{\Omega}\hat{\eta}] = \int_{\mathbb{R}} \hat{\rho}(.,0)\hat{\Theta}(.,0)\hat{\eta}(.,0) \,,$$

where we have used the Jacobian $\frac{\partial(x,t)}{\partial(\xi,\tau)} = \rho$. The last relation, which is true for any $\hat{\eta} \in D(\mathbb{R}\times\mathbb{R}_+)$, says that:

$$(\hat{\rho}\hat{\Theta})_\tau + (\hat{\rho}\hat{u}\hat{\Theta})_\xi - (\frac{\hat{\Theta}_\xi}{\hat{\rho}})_\xi = \hat{\rho}\hat{\Omega} ,$$

in the sense of the distributions in $\mathbb{R} \times \mathbb{R}_+$. ▪

To end the proof of Theorem 3.9, we still need the following lemma:

LEMMA 7.4:

There exists at most one *weak solution* $(\hat{\Theta},\hat{Y},\hat{\rho},\hat{u},\hat{p})$ of (3.4)-(3.5) satisfying:

$$\hat{u} \in C(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}) , \ \hat{\rho} \in C(\mathbb{R}\times\mathbb{R}_+,\mathbb{R}) . \blacksquare \tag{7.6}$$

PROOF: Let $(\hat{\Theta},\hat{Y},\hat{\rho},\hat{u},\hat{p})$ be a *weak solution* of (3.4)-(3.5). Thanks to (7.6), the transformations (7.3) and (7.4) can be used to show (exactly as in the proof of Proposition 7.3) that $(\hat{\Theta},\hat{Y},\hat{\rho},\hat{u},\hat{p})$ corresponds to a *weak solution* (Θ,Y,ρ,u,p) of (3.1)-(3.3); the uniqueness then follows from the Sections 5 and 6. ▪

REMARK 7.5: The uniqueness of a *weak solution* of (3.4)-(3.5) can also be proven without (7.6). In this case, the equivalence between Lagrangian locally bounded *weak solutions* and Eulerian locally bounded *weak solutions* still holds, but is less simple to prove (see [13]). ▪

8. EXTENSION TO CHEMICALLY COMPLEX FLAMES

In this section we extend our analysis to the equations of a chemically complex flame propagating in a dilute premixed gaseous mixture.

8.1. Physical assumptions

We will assume that the mixture is made up of N components A_1, A_2 ... A_N, whose mass fractions are respectively Y_1, Y_2 ... Y_N. The last specie A_N is chemically inert and the reactants and products are highly diluted in a bath of A_N :

$$\sum_{s=1}^{N-1} Y_s \ll Y_N . \tag{8.1}$$

It makes therefore sense to consider that the specific heat c_p and the thermal conductivity λ of the mixture are those of the inert. Also assuming that the matrix of the diffusion coefficients is diagonal (the diffusion flux for the s^{th} component only depends on ∇Y_s), we obtain that all the species have equal diffusivities (see [5], p. 8), a fairly classical assumption.

Let M be the number of irreversible chemical reactions taking place in the mixture. From $1 \leq r \leq M$, the r^{th} reaction can be written as:

$$\sum_{s=1}^{N-1} \nu_{rs} A_s \rightarrow \sum_{s=1}^{N-1} \mu_{rs} A_s .$$

where the stoechiometric coefficients ν_{rs} and μ_{rs} are positive integers [ν_{rs} (resp: μ_{rs}) is equal to zero if the specie A_s is not a reactant (resp: a product) in the r^{th} reaction]. Let ω_r be the rate at which this reaction proceeds (a relation analogous to (2.2) gives ω_r as a function of the temperature and the mass fractions Y_s).

We can now write the governing equations of the propagation of this chemically complex flame under the form:

$$\begin{cases} \rho_\tau + (\rho u)_\xi = 0 \,, \\ \rho u_\tau + \rho u u_\xi = -p_\xi \,, \\ \rho c_p T_\tau + \rho u c_p T_\xi - (\lambda T_\xi)_\xi = \sum_{r=1}^{M} Q_r \omega_r \,, \\ \rho (Y_s)_\tau + \rho u (Y_s)_\xi - (\rho D (Y_s)_\xi)_\xi = m_s \sum_{r=1}^{M} (\mu_{rs} - \nu_{rs}) \omega_r \quad \text{for } 1 \leq s \leq N \,, \\ \sum_{s=1}^{N} Y_s = 1 \,; \end{cases} \tag{8.2}$$

$$\rho T = m_N \frac{P}{R} \,. \tag{8.3}$$

We have defined $\nu_{rN} = \mu_{rN} = 0$ for all r. The heat released by the r^{th} reaction, which is no more assumed to be exothermic, is denoted by Q_r, and m_s is the molecular mass of the s^{th} specie; the other notations are defined as in Section 2.

REMARK 8.1: The form $\rho T = m_N \frac{P}{R}$ of the equation of state follows from the assumption (8.1). The perfect gas law gives the value $P_s = RT(\frac{\rho Y_s}{m_s})$ for the partial pressure of each specie. Using Dalton's law we get $P = \rho RT \sum_{s=1}^{N} \frac{Y_s}{m_s}$ for the total pressure P, and this last expression reduces to (8.3) in view of (8.1). •

Assuming again that the Lewis number $Le = \frac{\lambda}{\rho c_p D}$, the specific heat c_p and the ratio $\frac{\lambda}{T}$ are constant, we can write an Eulerian and a Lagrangian normalized form of (8.2)-(8.3) as follows:

$$\begin{cases} \rho_\tau + (\rho u)_\xi = 0 \,, \\ (\rho u)_\tau + (\rho u^2)_\xi = -p_\xi \,, \\ (\rho \Theta)_\tau + (\rho u \Theta)_\xi - \left(\frac{\Theta_\xi}{\rho}\right)_\xi = \sum_{r=1}^{M} Q_r \rho \Omega_r \,, \\ (\rho Y_s)_\tau + (\rho u Y_s)_\xi - \frac{1}{Le} \left[\frac{(Y_s)_\xi}{\rho} \right]_\xi = m_s \sum_{r=1}^{M} (\mu_{rs} - \nu_{rs}) \rho \Omega_r \quad \text{for } 1 \leq s \leq N \,, \\ \sum_{s=1}^{N} Y_s = 1 \,, \\ (\Theta + \alpha) \rho = 1 \,. \end{cases} \tag{8.4}$$

$$\left\{\begin{array}{l} \Theta_t - \Theta_{xx} = \sum_{r=1}^{M} Q_r \Omega_r \,, \\ (Y_s)_t - \dfrac{(Y_s)_{xx}}{Le} = m_s \sum_{r=1}^{M} (\mu_{rs} - \nu_{rs})\Omega_r \quad \text{for } 1 \leq s \leq N \,, \\ \sum_{s=1}^{N} Y_s = 1 \,, \\ (\Theta + \alpha)\rho = 1 \,, \\ u_x = \Theta_t \,, \\ u_t + p_x = 0 \,. \end{array}\right. \tag{8.5}$$

The following boundary conditions are associated to the above systems:

$$\left\{\begin{array}{l} \Theta(-\infty) = 0 \,,\ \Theta(+\infty) = 1 \,, \\ Y_s(-\infty) = Y_{su} \,,\ Y_s(+\infty) = Y_{sb} \,, \\ u(-\infty) = u^0 \,,\ p(-\infty) = 0 \,. \end{array}\right.$$

Let us denote $\Omega_r = \prod_{s=1}^{N-1} Y_s^{\nu_{rs}} f_r(\Theta)$. Since we use time-independent boundary conditions, we have to assume that the two thermo-chemical states prescribed at the boundaries $-\infty$ and $+\infty$ correspond to equilibria, i.e:

$$\forall\, r \in \{1,2 \cdots M\} \,,\ f_r(0) = 0 \,;$$

and:

$$\forall\, r \in \{1,2 \cdots M\} \,,\ \prod_{s=1}^{N-1} Y_{sb}^{\nu_{rs}} = 0 \,.$$

(all the reactant concentrations vanish in the burnt state).

It is then straightforward to extend to the systems (8.4) and (8.5) the results stated in Section 3 (with a change of unknowns similar to (5.3) and assumptions analogous to (3.9)-(3.15)). Stating in detail the hypotheses and the theorems would be too long but no new difficulty appears for applying the arguments of Sections 5, 6 and 7 to the systems (8.4) and (8.5).

REMARK 8.2: The global existence and uniqueness results stated in Section 3 are also easily extended to the case of the non adiabatic propagation of a planar flame. In this case, the energy balance equation (2.3.b) becomes:

$$\rho c_p T_\tau + \rho u c_p T_\xi - (\lambda T_\xi)_\xi = Q\,\omega(Y,T) - \kappa(T) \,,$$

where $\kappa(T) \geqq 0$ represents the heat losses (see [5]; for instance $\kappa(T) \equiv k(T - T_{ref})$ if only conductive heat losses are considered). In Lagrangian coordinates, the energy equation (2.10.a) reads as:

$$\Theta_t = \Theta_{xx} + \Omega(Y,\Theta) - \hat{\kappa}(\Theta) ,$$

with $\hat{\kappa}(\Theta) \geq 0$, $\hat{\kappa}(0) = 0$, and the arguments presented in Sections 5 and 6 apply.

■

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